

# Multiplicative-additive linear logic (MALL)

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# 1 Measure of derivations

- Case(s) rule 1

$$\frac{}{\bullet h_1 : \vdash 1} 1 \rightarrow \frac{}{\bullet \bullet h_1 : \vdash 1} 1$$

- Case(s) rule \$

$$\frac{h_1 : \vdash F_3, F_4, \Delta_2}{\bullet h_1 : \vdash \Delta_2, F_3 \S F_4} \$ \rightarrow \frac{\frac{h_1 : \vdash \Delta_2, F_3, F_4}{\bullet h_1 : \vdash \Delta_2, F_3, F_4} \text{IH}}{\bullet \bullet h_1 : \vdash \Delta_2, F_3 \S F_4} \text{ax} \$$$

- Case(s) rule &

$$\frac{h_1 : \vdash F_3, \Delta_2 \quad h_1 : \vdash F_4, \Delta_2}{\bullet h_1 : \vdash \Delta_2, F_3 \& F_4} \& \rightarrow \frac{\frac{h_1 : \vdash \Delta_2, F_3}{\bullet h_1 : \vdash \Delta_2, F_3} \text{IH} \quad \frac{h_1 : \vdash \Delta_2, F_4}{\bullet h_1 : \vdash \Delta_2, F_4} \text{IH}}{\bullet \bullet h_1 : \vdash \Delta_2, F_3 \& F_4} \text{ax} \&$$

- Case(s) rule  $\oplus_B$

$$\frac{h_1 : \vdash F_4, \Delta_2}{\bullet h_1 : \vdash \Delta_2, F_3 \oplus F_4} \oplus_B \rightarrow \frac{\frac{h_1 : \vdash \Delta_2, F_4}{\bullet h_1 : \vdash \Delta_2, F_4} \text{IH}}{\bullet \bullet h_1 : \vdash \Delta_2, F_3 \oplus F_4} \text{ax} \oplus_B$$

- Case(s) rule  $\oplus_A$

$$\frac{h_1 : \vdash F_3, \Delta_2}{\bullet h_1 : \vdash \Delta_2, F_3 \oplus F_4} \oplus_A \rightarrow \frac{\frac{h_1 : \vdash \Delta_2, F_3}{\bullet h_1 : \vdash \Delta_2, F_3} \text{IH}}{\bullet \bullet h_1 : \vdash \Delta_2, F_3 \oplus F_4} \text{ax} \oplus_A$$

- Case(s) rule  $\perp$

$$\frac{h_1 : \vdash \Delta_2}{\bullet h_1 : \vdash \perp, \Delta_2} \perp \rightarrow \frac{\frac{h_1 : \vdash \Delta_2}{\bullet h_1 : \vdash \Delta_2} \text{IH}}{\bullet \bullet h_1 : \vdash \Delta_2, \perp} \text{ax} \perp$$

- Case(s) rule  $\top$

$$\frac{}{\bullet h_1 : \vdash \top, \Delta_2} \top \rightarrow \frac{}{\bullet \bullet h_1 : \vdash \Delta_2, \top} \top$$

- Case(s) rule  $I$

$$\frac{}{\bullet h_1 : \vdash p(n_2), \wedge(n_2)} I \rightarrow \frac{}{\bullet \bullet h_1 : \vdash p(n_2), \wedge(n_2)} I$$

- Case(s) rule  $\otimes$

$$\frac{h_1 : \vdash F_4, \Delta_2 \quad h_1 : \vdash F_5, \Delta_3}{\bullet h_1 : \vdash \Delta_2, \Delta_3, F_4 \otimes F_5} \otimes \rightarrow \frac{\frac{h_1 : \vdash \Delta_2, F_4}{\bullet h_1 : \vdash \Delta_2, F_4} \text{IH} \quad \frac{h_1 : \vdash \Delta_3, F_5}{\bullet h_1 : \vdash \Delta_3, F_5} \text{IH}}{\bullet \bullet h_1 : \vdash \Delta_2, \Delta_3, F_4 \otimes F_5} \text{ax} \otimes$$

## 2 Invertibility of Rules

### 2.1 Status of $\mathbf{1}$ : : Invertible

- Case rule  $\mathbf{1}$

$$\frac{}{\bullet h_1 : \vdash \mathbf{1}} \mathbf{1} \rightarrow \text{trivial}$$

- Case rule  $\$$
- Case rule  $\&$
- Case rule  $\oplus_B$
- Case rule  $\oplus_A$
- Case rule  $\perp$
- Case rule  $\top$
- Case rule  $I$
- Case rule  $\otimes$

### 2.2 Status of $\$$ : : Invertible

- Case rule  $\mathbf{1}$
- Case rule  $\$$

$$\frac{h_3 : \vdash F_4, F_5, \Delta_6, F_1 \$F_2}{\bullet h_3 : \vdash (\Delta_6, F_1 \$F_2), F_4 \$F_5} \$ \rightarrow \frac{}{h_3 : \vdash \Delta_6, F_1, F_2, F_4, F_5} \text{ax/ind} \frac{}{\bullet h_3 : \vdash \Delta_6, F_1, F_2, F_4 \$F_5} \$$$

$$\frac{h_1 : \vdash F_3, F_4, \Delta_2}{\bullet h_1 : \vdash \Delta_2, F_3 \$F_4} \$ \rightarrow \frac{}{h_1 : \vdash \Delta_2, F_3, F_4} \text{ax} \frac{}{\bullet h_1 : \vdash \Delta_2, F_3, F_4} \text{height}$$

- Case rule  $\&$

$$\frac{h_3 : \vdash F_4, \Delta_6, F_1 \$F_2 \quad h_3 : \vdash F_5, \Delta_6, F_1 \$F_2}{\bullet h_3 : \vdash (\Delta_6, F_1 \$F_2), F_4 \&F_5} \& \rightarrow \frac{}{h_3 : \vdash \Delta_6, F_1, F_2, F_4} \text{ax/ind} \frac{}{h_3 : \vdash \Delta_6, F_1, F_2, F_5} \text{ax/ind} \frac{}{\bullet h_3 : \vdash \Delta_6, F_1, F_2, F_4 \&F_5} \&$$

- Case rule  $\oplus_B$

$$\frac{h_3 : \vdash F_5, \Delta_6, F_1 \$F_2}{\bullet h_3 : \vdash (\Delta_6, F_1 \$F_2), F_4 \oplus F_5} \oplus_B \rightarrow \frac{}{h_3 : \vdash \Delta_6, F_1, F_2, F_5} \text{ax/ind} \frac{}{\bullet h_3 : \vdash \Delta_6, F_1, F_2, F_4 \oplus F_5} \oplus_B$$

- Case rule  $\oplus_A$

$$\frac{h_3 \vdash F_4, \Delta_6, F_1 \S F_2}{\bullet h_3 \vdash (\Delta_6, F_1 \S F_2), F_4 \oplus F_5} \oplus_A \rightarrow \frac{\overline{h_3 \vdash \Delta_6, F_1, F_2, F_4} \text{ ax/ind}}{\bullet h_3 \vdash \Delta_6, F_1, F_2, F_4 \oplus F_5} \oplus_A$$

- Case rule  $\perp$

$$\frac{h_3 \vdash \Delta_4, F_1 \S F_2}{\bullet h_3 \vdash \perp, \Delta_4, F_1 \S F_2} \perp \rightarrow \frac{\overline{h_3 \vdash \Delta_4, F_1, F_2} \text{ ax/ind}}{\bullet h_3 \vdash \Delta_4, F_1, F_2, \perp} \perp$$

- Case rule  $\top$

$$\overline{\bullet h_3 \vdash \top, \Delta_4, F_1 \S F_2} \top \rightarrow \overline{\bullet h_3 \vdash \Delta_4, F_1, F_2, \top} \top$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{h_3 \vdash F_5, \Delta_7, F_1 \S F_2 \quad h_3 \vdash F_6, \Delta_4}{\bullet h_3 \vdash (\Delta_7, F_1 \S F_2), \Delta_4, F_5 \otimes F_6} \otimes \rightarrow \frac{\overline{h_3 \vdash \Delta_7, F_1, F_2, F_5} \text{ ax/ind} \quad \overline{h_3 \vdash \Delta_4, F_6} \text{ ax}}{\bullet h_3 \vdash \Delta_4, \Delta_7, F_1, F_2, F_5 \otimes F_6} \otimes$$

$$\frac{h_3 \vdash F_5, \Delta_4 \quad h_3 \vdash F_6, \Delta_7, F_1 \S F_2}{\bullet h_3 \vdash \Delta_4, (\Delta_7, F_1 \S F_2), F_5 \otimes F_6} \otimes \rightarrow \frac{\overline{h_3 \vdash \Delta_4, F_5} \text{ ax} \quad \overline{h_3 \vdash \Delta_7, F_1, F_2, F_6} \text{ ax/ind}}{\bullet h_3 \vdash \Delta_4, \Delta_7, F_1, F_2, F_5 \otimes F_6} \otimes$$

## 2.3 Status of $\&$ : (Left Premise): Invertible

- Case rule  $1$

- Case rule  $\S$

$$\frac{h_3 \vdash F_4, F_5, \Delta_6, F_1 \& F_2}{\bullet h_3 \vdash (\Delta_6, F_1 \& F_2), F_4 \S F_5} \S \rightarrow \frac{\overline{h_3 \vdash \Delta_6, F_1, F_4, F_5} \text{ ax/ind}}{\bullet h_3 \vdash \Delta_6, F_1, F_4 \S F_5} \S$$

- Case rule  $\&$

$$\frac{h_3 \vdash F_4, \Delta_6, F_1 \& F_2 \quad h_3 \vdash F_5, \Delta_6, F_1 \& F_2}{\bullet h_3 \vdash (\Delta_6, F_1 \& F_2), F_4 \& F_5} \& \rightarrow \frac{\overline{h_3 \vdash \Delta_6, F_1, F_4} \text{ ax/ind} \quad \overline{h_3 \vdash \Delta_6, F_1, F_5} \text{ ax/ind}}{\bullet h_3 \vdash \Delta_6, F_1, F_4 \& F_5} \&$$

$$\frac{h_1 \vdash F_3, \Delta_2 \quad h_1 \vdash F_4, \Delta_2}{\bullet h_1 \vdash \Delta_2, F_3 \& F_4} \& \rightarrow \frac{\overline{h_1 \vdash \Delta_2, F_3} \text{ ax}}{\bullet h_1 \vdash \Delta_2, F_3} \text{ height}$$

- Case rule  $\oplus_B$

$$\frac{h_3 \vdash F_5, \Delta_6, F_1 \& F_2}{\bullet h_3 \vdash (\Delta_6, F_1 \& F_2), F_4 \oplus F_5} \oplus_B \rightarrow \frac{\overline{h_3 \vdash \Delta_6, F_1, F_5} \text{ ax/ind}}{\bullet h_3 \vdash \Delta_6, F_1, F_4 \oplus F_5} \oplus_B$$

- Case rule  $\oplus_A$

$$\frac{h_3 \text{ :- } F_4, \Delta_6, F_1 \& F_2}{\bullet h_3 \text{ :- } (\Delta_6, F_1 \& F_2), F_4 \oplus F_5} \oplus_A \rightarrow \frac{\overline{h_3 \text{ :- } \Delta_6, F_1, F_4}}{\bullet h_3 \text{ :- } \Delta_6, F_1, F_4 \oplus F_5} \text{ ax/ind} \oplus_A$$

- Case rule  $\perp$

$$\frac{h_3 \text{ :- } \Delta_4, F_1 \& F_2}{\bullet h_3 \text{ :- } \perp, \Delta_4, F_1 \& F_2} \perp \rightarrow \frac{\overline{h_3 \text{ :- } \Delta_4, F_1}}{\bullet h_3 \text{ :- } \Delta_4, F_1, \perp} \text{ ax/ind} \perp$$

- Case rule  $\top$

$$\overline{\bullet h_3 \text{ :- } \top, \Delta_4, F_1 \& F_2} \top \rightarrow \overline{\bullet h_3 \text{ :- } \Delta_4, F_1, \top} \top$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{h_3 \text{ :- } F_5, \Delta_7, F_1 \& F_2 \quad h_3 \text{ :- } F_6, \Delta_4}{\bullet h_3 \text{ :- } (\Delta_7, F_1 \& F_2), \Delta_4, F_5 \otimes F_6} \otimes \rightarrow \frac{\overline{h_3 \text{ :- } \Delta_7, F_1, F_5} \text{ ax/ind} \quad \overline{h_3 \text{ :- } \Delta_4, F_6} \text{ ax}}{\bullet h_3 \text{ :- } \Delta_4, \Delta_7, F_1, F_5 \otimes F_6} \otimes$$

$$\frac{h_3 \text{ :- } F_5, \Delta_4 \quad h_3 \text{ :- } F_6, \Delta_7, F_1 \& F_2}{\bullet h_3 \text{ :- } \Delta_4, (\Delta_7, F_1 \& F_2), F_5 \otimes F_6} \otimes \rightarrow \frac{\overline{h_3 \text{ :- } \Delta_4, F_5} \text{ ax} \quad \overline{h_3 \text{ :- } \Delta_7, F_1, F_6} \text{ ax/ind}}{\bullet h_3 \text{ :- } \Delta_4, \Delta_7, F_1, F_5 \otimes F_6} \otimes$$

## 2.4 Status of $\&$ (Right Premise): : Invertible

- Case rule 1

- Case rule  $\$$

$$\frac{h_3 \text{ :- } F_4, F_5, \Delta_6, F_1 \& F_2}{\bullet h_3 \text{ :- } (\Delta_6, F_1 \& F_2), F_4 \$ F_5} \$ \rightarrow \frac{\overline{h_3 \text{ :- } \Delta_6, F_2, F_4, F_5} \text{ ax/ind}}{\bullet h_3 \text{ :- } \Delta_6, F_2, F_4 \$ F_5} \$$$

- Case rule  $\&$

$$\frac{h_3 \text{ :- } F_4, \Delta_6, F_1 \& F_2 \quad h_3 \text{ :- } F_5, \Delta_6, F_1 \& F_2}{\bullet h_3 \text{ :- } (\Delta_6, F_1 \& F_2), F_4 \& F_5} \& \rightarrow \frac{\overline{h_3 \text{ :- } \Delta_6, F_2, F_4} \text{ ax/ind} \quad \overline{h_3 \text{ :- } \Delta_6, F_2, F_5} \text{ ax/ind}}{\bullet h_3 \text{ :- } \Delta_6, F_2, F_4 \& F_5} \&$$

$$\frac{h_1 \text{ :- } F_3, \Delta_2 \quad h_1 \text{ :- } F_4, \Delta_2}{\bullet h_1 \text{ :- } \Delta_2, F_3 \& F_4} \& \rightarrow \frac{\overline{h_1 \text{ :- } \Delta_2, F_4} \text{ ax}}{\bullet h_1 \text{ :- } \Delta_2, F_4} \text{ height}$$

- Case rule  $\oplus_B$

$$\frac{h_3 \text{ :- } F_5, \Delta_6, F_1 \& F_2}{\bullet h_3 \text{ :- } (\Delta_6, F_1 \& F_2), F_4 \oplus F_5} \oplus_B \rightarrow \frac{\overline{h_3 \text{ :- } \Delta_6, F_2, F_5} \text{ ax/ind}}{\bullet h_3 \text{ :- } \Delta_6, F_2, F_4 \oplus F_5} \oplus_B$$

- Case rule  $\oplus_A$

$$\frac{h_3 : \vdash F_4, \Delta_6, F_1 \& F_2}{\bullet h_3 : \vdash (\Delta_6, F_1 \& F_2), F_4 \oplus F_5} \oplus_A \rightarrow \frac{\overline{h_3 : \vdash \Delta_6, F_2, F_4} \text{ ax/ind}}{\bullet h_3 : \vdash \Delta_6, F_2, F_4 \oplus F_5} \oplus_A$$

- Case rule  $\perp$

$$\frac{h_3 : \vdash \Delta_4, F_1 \& F_2}{\bullet h_3 : \vdash \perp, \Delta_4, F_1 \& F_2} \perp \rightarrow \frac{\overline{h_3 : \vdash \Delta_4, F_2} \text{ ax/ind}}{\bullet h_3 : \vdash \Delta_4, F_2, \perp} \perp$$

- Case rule  $\top$

$$\overline{\bullet h_3 : \vdash \top, \Delta_4, F_1 \& F_2} \top \rightarrow \overline{\bullet h_3 : \vdash \Delta_4, F_2, \top} \top$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{h_3 : \vdash F_5, \Delta_7, F_1 \& F_2 \quad h_3 : \vdash F_6, \Delta_4}{\bullet h_3 : \vdash (\Delta_7, F_1 \& F_2), \Delta_4, F_5 \otimes F_6} \otimes \rightarrow \frac{\overline{h_3 : \vdash \Delta_7, F_2, F_5} \text{ ax/ind} \quad \overline{h_3 : \vdash \Delta_4, F_6} \text{ ax}}{\bullet h_3 : \vdash \Delta_4, \Delta_7, F_2, F_5 \otimes F_6} \otimes$$

$$\frac{h_3 : \vdash F_5, \Delta_4 \quad h_3 : \vdash F_6, \Delta_7, F_1 \& F_2}{\bullet h_3 : \vdash \Delta_4, (\Delta_7, F_1 \& F_2), F_5 \otimes F_6} \otimes \rightarrow \frac{\overline{h_3 : \vdash \Delta_4, F_5} \text{ ax} \quad \overline{h_3 : \vdash \Delta_7, F_2, F_6} \text{ ax/ind}}{\bullet h_3 : \vdash \Delta_4, \Delta_7, F_2, F_5 \otimes F_6} \otimes$$

## 2.5 Status of $\oplus_B$ : : Non invertible

- Case rule  $\mathbf{1}$

- Case rule  $\$$

$$\frac{h_3 : \vdash F_4, F_5, \Delta_6, F_1 \oplus F_2}{\bullet h_3 : \vdash (\Delta_6, F_1 \oplus F_2), F_4 \$ F_5} \$ \rightarrow \frac{\overline{h_3 : \vdash \Delta_6, F_2, F_4, F_5} \text{ ax/ind}}{\bullet h_3 : \vdash \Delta_6, F_2, F_4 \$ F_5} \$$$

- Case rule  $\&$

$$\frac{h_3 : \vdash F_4, \Delta_6, F_1 \oplus F_2 \quad h_3 : \vdash F_5, \Delta_6, F_1 \oplus F_2}{\bullet h_3 : \vdash (\Delta_6, F_1 \oplus F_2), F_4 \& F_5} \& \rightarrow \frac{\overline{h_3 : \vdash \Delta_6, F_2, F_4} \text{ ax/ind} \quad \overline{h_3 : \vdash \Delta_6, F_2, F_5} \text{ ax/ind}}{\bullet h_3 : \vdash \Delta_6, F_2, F_4 \& F_5} \&$$

- Case rule  $\oplus_B$

$$\frac{h_3 : \vdash F_5, \Delta_6, F_1 \oplus F_2}{\bullet h_3 : \vdash (\Delta_6, F_1 \oplus F_2), F_4 \oplus F_5} \oplus_B \rightarrow \frac{\overline{h_3 : \vdash \Delta_6, F_2, F_5} \text{ ax/ind}}{\bullet h_3 : \vdash \Delta_6, F_2, F_4 \oplus F_5} \oplus_B$$

$$\frac{h_1 : \vdash F_4, \Delta_2}{\bullet h_1 : \vdash \Delta_2, F_3 \oplus F_4} \oplus_B \rightarrow \frac{\overline{h_1 : \vdash \Delta_2, F_4} \text{ ax}}{\bullet h_1 : \vdash \Delta_2, F_4} \text{ height}$$

- Case rule  $\oplus_A$

$$\frac{h_3 : \vdash F_4, \Delta_6, F_1 \oplus F_2}{\bullet h_3 : \vdash (\Delta_6, F_1 \oplus F_2), F_4 \oplus F_5} \oplus_A \rightarrow \frac{\overline{h_3 : \vdash \Delta_6, F_2, F_4} \text{ ax/ind}}{\bullet h_3 : \vdash \Delta_6, F_2, F_4 \oplus F_5} \oplus_A$$

$$\frac{h_1 : \vdash F_3, \Delta_2}{\bullet h_1 : \vdash \Delta_2, F_3 \oplus F_4} \oplus_A \rightarrow \overline{\bullet h_1 : \vdash F_4, \Delta_2} \text{ fail}$$

- Case rule  $\perp$

$$\frac{h_3 : \vdash \Delta_4, F_1 \oplus F_2}{\bullet h_3 : \vdash \perp, \Delta_4, F_1 \oplus F_2} \perp \rightarrow \frac{\overline{h_3 : \vdash \Delta_4, F_2} \text{ ax/ind}}{\bullet h_3 : \vdash \Delta_4, F_2, \perp} \perp$$

- Case rule  $\top$

$$\overline{\bullet h_3 : \vdash \top, \Delta_4, F_1 \oplus F_2} \top \rightarrow \overline{\bullet h_3 : \vdash \Delta_4, F_2, \top} \top$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{h_3 : \vdash F_5, \Delta_7, F_1 \oplus F_2 \quad h_3 : \vdash F_6, \Delta_4}{\bullet h_3 : \vdash (\Delta_7, F_1 \oplus F_2), \Delta_4, F_5 \otimes F_6} \otimes \rightarrow \frac{\overline{h_3 : \vdash \Delta_7, F_2, F_5} \text{ ax/ind} \quad \overline{h_3 : \vdash \Delta_4, F_6} \text{ ax}}{\bullet h_3 : \vdash \Delta_4, \Delta_7, F_2, F_5 \otimes F_6} \otimes$$

$$\frac{h_3 : \vdash F_5, \Delta_4 \quad h_3 : \vdash F_6, \Delta_7, F_1 \oplus F_2}{\bullet h_3 : \vdash \Delta_4, (\Delta_7, F_1 \oplus F_2), F_5 \otimes F_6} \otimes \rightarrow \frac{\overline{h_3 : \vdash \Delta_4, F_5} \text{ ax} \quad \overline{h_3 : \vdash \Delta_7, F_2, F_6} \text{ ax/ind}}{\bullet h_3 : \vdash \Delta_4, \Delta_7, F_2, F_5 \otimes F_6} \otimes$$

## 2.6 Status of $\oplus_A$ : : Non invertible

- Case rule  $1$

- Case rule  $\$$

$$\frac{h_3 : \vdash F_4, F_5, \Delta_6, F_1 \oplus F_2}{\bullet h_3 : \vdash (\Delta_6, F_1 \oplus F_2), F_4 \$ F_5} \$ \rightarrow \frac{\overline{h_3 : \vdash \Delta_6, F_1, F_4, F_5} \text{ ax/ind}}{\bullet h_3 : \vdash \Delta_6, F_1, F_4 \$ F_5} \$$$

- Case rule  $\&$

$$\frac{h_3 : \vdash F_4, \Delta_6, F_1 \oplus F_2 \quad h_3 : \vdash F_5, \Delta_6, F_1 \oplus F_2}{\bullet h_3 : \vdash (\Delta_6, F_1 \oplus F_2), F_4 \& F_5} \& \rightarrow \frac{\overline{h_3 : \vdash \Delta_6, F_1, F_4} \text{ ax/ind} \quad \overline{h_3 : \vdash \Delta_6, F_1, F_5} \text{ ax/ind}}{\bullet h_3 : \vdash \Delta_6, F_1, F_4 \& F_5} \&$$

- Case rule  $\oplus_B$

$$\frac{h_3 : \vdash F_5, \Delta_6, F_1 \oplus F_2}{\bullet h_3 : \vdash (\Delta_6, F_1 \oplus F_2), F_4 \oplus F_5} \oplus_B \rightarrow \frac{\overline{h_3 : \vdash \Delta_6, F_1, F_5} \text{ ax/ind}}{\bullet h_3 : \vdash \Delta_6, F_1, F_4 \oplus F_5} \oplus_B$$

$$\frac{h_1 : \vdash F_4, \Delta_2}{\bullet h_1 : \vdash \Delta_2, F_3 \oplus F_4} \oplus_B \rightarrow \overline{\bullet h_1 : \vdash F_3, \Delta_2} \text{ fail}$$

- Case rule  $\oplus_A$

$$\frac{h_3 \text{ :- } F_4, \Delta_6, F_1 \oplus F_2}{\bullet h_3 \text{ :- } (\Delta_6, F_1 \oplus F_2), F_4 \oplus F_5} \oplus_A \rightarrow \frac{\overline{h_3 \text{ :- } \Delta_6, F_1, F_4} \text{ ax/ind}}{\bullet h_3 \text{ :- } \Delta_6, F_1, F_4 \oplus F_5} \oplus_A$$

$$\frac{h_1 \text{ :- } F_3, \Delta_2}{\bullet h_1 \text{ :- } \Delta_2, F_3 \oplus F_4} \oplus_A \rightarrow \frac{\overline{h_1 \text{ :- } \Delta_2, F_3} \text{ ax}}{\bullet h_1 \text{ :- } \Delta_2, F_3} \text{ height}$$

- Case rule  $\perp$

$$\frac{h_3 \text{ :- } \Delta_4, F_1 \oplus F_2}{\bullet h_3 \text{ :- } \perp, \Delta_4, F_1 \oplus F_2} \perp \rightarrow \frac{\overline{h_3 \text{ :- } \Delta_4, F_1} \text{ ax/ind}}{\bullet h_3 \text{ :- } \Delta_4, F_1, \perp} \perp$$

- Case rule  $\top$

$$\overline{\bullet h_3 \text{ :- } \top, \Delta_4, F_1 \oplus F_2} \top \rightarrow \overline{\bullet h_3 \text{ :- } \Delta_4, F_1, \top} \top$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{h_3 \text{ :- } F_5, \Delta_7, F_1 \oplus F_2 \quad h_3 \text{ :- } F_6, \Delta_4}{\bullet h_3 \text{ :- } (\Delta_7, F_1 \oplus F_2), \Delta_4, F_5 \otimes F_6} \otimes \rightarrow \frac{\overline{h_3 \text{ :- } \Delta_7, F_1, F_5} \text{ ax/ind} \quad \overline{h_3 \text{ :- } \Delta_4, F_6} \text{ ax}}{\bullet h_3 \text{ :- } \Delta_4, \Delta_7, F_1, F_5 \otimes F_6} \otimes$$

$$\frac{h_3 \text{ :- } F_5, \Delta_4 \quad h_3 \text{ :- } F_6, \Delta_7, F_1 \oplus F_2}{\bullet h_3 \text{ :- } \Delta_4, (\Delta_7, F_1 \oplus F_2), F_5 \otimes F_6} \otimes \rightarrow \frac{\overline{h_3 \text{ :- } \Delta_4, F_5} \text{ ax} \quad \overline{h_3 \text{ :- } \Delta_7, F_1, F_6} \text{ ax/ind}}{\bullet h_3 \text{ :- } \Delta_4, \Delta_7, F_1, F_5 \otimes F_6} \otimes$$

## 2.7 Status of $\perp$ : : Invertible

- Case rule  $\mathbf{1}$

- Case rule  $\$$

$$\frac{h_1 \text{ :- } \perp, F_2, F_3, \Delta_4}{\bullet h_1 \text{ :- } (\perp, \Delta_4), F_2 \$ F_3} \$ \rightarrow \frac{\overline{h_1 \text{ :- } \Delta_4, F_2, F_3} \text{ ax/ind}}{\bullet h_1 \text{ :- } \Delta_4, F_2 \$ F_3} \$$$

- Case rule  $\&$

$$\frac{h_1 \text{ :- } \perp, F_2, \Delta_4 \quad h_1 \text{ :- } \perp, F_3, \Delta_4}{\bullet h_1 \text{ :- } (\perp, \Delta_4), F_2 \& F_3} \& \rightarrow \frac{\overline{h_1 \text{ :- } \Delta_4, F_2} \text{ ax/ind} \quad \overline{h_1 \text{ :- } \Delta_4, F_3} \text{ ax/ind}}{\bullet h_1 \text{ :- } \Delta_4, F_2 \& F_3} \&$$

- Case rule  $\oplus_B$

$$\frac{h_1 \text{ :- } \perp, F_3, \Delta_4}{\bullet h_1 \text{ :- } (\perp, \Delta_4), F_2 \oplus F_3} \oplus_B \rightarrow \frac{\overline{h_1 \text{ :- } \Delta_4, F_3} \text{ ax/ind}}{\bullet h_1 \text{ :- } \Delta_4, F_2 \oplus F_3} \oplus_B$$



- Case rule  $\oplus_A$

$$\frac{h_1 : \vdash \perp, F_2, \Delta_4}{\bullet h_1 : \vdash (\perp, \Delta_4), F_2 \oplus F_3} \oplus_A \rightarrow \frac{\overline{h_1 : \vdash \Delta_4, F_2} \text{ ax/ind}}{\bullet h_1 : \vdash \Delta_4, F_2 \oplus F_3} \oplus_A$$

- Case rule  $\perp$

$$\frac{h_1 : \vdash \Delta_2}{\bullet h_1 : \vdash \perp, \Delta_2} \perp \rightarrow \frac{\overline{h_1 : \vdash \Delta_2} \text{ ax}}{\bullet h_1 : \vdash \Delta_2} \text{ height}$$

- Case rule  $\top$

$$\overline{\bullet h_1 : \vdash \top, \perp, \Delta_2} \top \rightarrow \overline{\bullet h_1 : \vdash \Delta_2, \top} \top$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{h_1 : \vdash \perp, F_3, \Delta_5 \quad h_1 : \vdash F_4, \Delta_2}{\bullet h_1 : \vdash (\perp, \Delta_5), \Delta_2, F_3 \otimes F_4} \otimes \rightarrow \frac{\overline{h_1 : \vdash \Delta_5, F_3} \text{ ax/ind} \quad \overline{h_1 : \vdash \Delta_2, F_4} \text{ ax}}{\bullet h_1 : \vdash \Delta_2, \Delta_5, F_3 \otimes F_4} \otimes$$

$$\frac{h_1 : \vdash F_3, \Delta_2 \quad h_1 : \vdash \perp, F_4, \Delta_5}{\bullet h_1 : \vdash \Delta_2, (\perp, \Delta_5), F_3 \otimes F_4} \otimes \rightarrow \frac{\overline{h_1 : \vdash \Delta_2, F_3} \text{ ax} \quad \overline{h_1 : \vdash \Delta_5, F_4} \text{ ax/ind}}{\bullet h_1 : \vdash \Delta_2, \Delta_5, F_3 \otimes F_4} \otimes$$

## 2.8 Status of $\top$ : : Invertible

- Case rule  $\mathbf{1}$

- Case rule  $\$$

$$\frac{h_1 : \vdash \top, F_2, F_3, \Delta_4}{\bullet h_1 : \vdash (\top, \Delta_4), F_2 \$ F_3} \$ \rightarrow \text{trivial}$$

- Case rule  $\&$

$$\frac{h_1 : \vdash \top, F_2, \Delta_4 \quad h_1 : \vdash \top, F_3, \Delta_4}{\bullet h_1 : \vdash (\top, \Delta_4), F_2 \& F_3} \& \rightarrow \text{trivial}$$

- Case rule  $\oplus_B$

$$\frac{h_1 : \vdash \top, F_3, \Delta_4}{\bullet h_1 : \vdash (\top, \Delta_4), F_2 \oplus F_3} \oplus_B \rightarrow \text{trivial}$$

- Case rule  $\oplus_A$

$$\frac{h_1 : \vdash \top, F_2, \Delta_4}{\bullet h_1 : \vdash (\top, \Delta_4), F_2 \oplus F_3} \oplus_A \rightarrow \text{trivial}$$

- Case rule  $\perp$

$$\frac{h_1 : \vdash \top, \Delta_2}{\bullet h_1 : \vdash \perp, \top, \Delta_2} \perp \rightarrow \text{trivial}$$

- Case rule  $\top$

$$\frac{}{\bullet h_1 : \vdash \top, \Delta_2} \top \rightarrow \text{trivial}$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{h_1 : \vdash \top, F_3, \Delta_5 \quad h_1 : \vdash F_4, \Delta_2}{\bullet h_1 : \vdash (\top, \Delta_5), \Delta_2, F_3 \otimes F_4} \otimes \rightarrow \text{trivial}$$

$$\frac{h_1 : \vdash F_3, \Delta_2 \quad h_1 : \vdash \top, F_4, \Delta_5}{\bullet h_1 : \vdash \Delta_2, (\top, \Delta_5), F_3 \otimes F_4} \otimes \rightarrow \text{trivial}$$

## 2.9 Status of $I$ : : Invertible

- Case rule  $\mathbf{1}$
- Case rule  $\$$
- Case rule  $\&$
- Case rule  $\oplus_B$
- Case rule  $\oplus_A$
- Case rule  $\perp$
- Case rule  $\top$
- Case rule  $I$

$$\frac{}{\bullet h_1 : \vdash p(n_2), \sim(n_2)} I \rightarrow \text{trivial}$$

- Case rule  $\otimes$

## 2.10 Status of $\otimes$ : (Left Premise): Non invertible

- Case rule  $\mathbf{1}$

- Case rule  $\$$

$$\frac{h_4 \text{ :- } F_5, F_6, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \$ F_6} \$ \rightarrow \frac{\overline{h_4 \text{ :- } \Delta_7, F_2, F_5, F_6} \text{ ax/ind}}{\bullet h_4 \text{ :- } \Delta_7, F_2, F_5 \$ F_6} \$$$

$$\frac{h_4 \text{ :- } F_5, F_6, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \$ F_6} \$ \rightarrow \frac{\overline{h_4 \text{ :- } \Delta_1, F_2} \text{ ax/ind}}{\bullet h_4 \text{ :- } \Delta_1, F_2} \text{ height}$$

- Case rule  $\&$

$$\frac{h_4 \text{ :- } F_5, \Delta_1, \Delta_7, F_2 \otimes F_3 \quad h_4 \text{ :- } F_6, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \& F_6} \& \rightarrow \frac{\overline{h_4 \text{ :- } \Delta_7, F_2, F_5} \text{ ax/ind} \quad \overline{h_4 \text{ :- } \Delta_7, F_2, F_6} \text{ ax/ind}}{\bullet h_4 \text{ :- } \Delta_7, F_2, F_5 \& F_6} \&$$

$$\frac{h_4 \text{ :- } F_5, \Delta_1, \Delta_7, F_2 \otimes F_3 \quad h_4 \text{ :- } F_6, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \& F_6} \& \rightarrow \frac{\overline{h_4 \text{ :- } \Delta_1, F_2} \text{ ax/ind}}{\bullet h_4 \text{ :- } \Delta_1, F_2} \text{ height}$$

- Case rule  $\oplus_B$

$$\frac{h_4 \text{ :- } F_6, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \oplus F_6} \oplus_B \rightarrow \frac{\overline{h_4 \text{ :- } \Delta_7, F_2, F_6} \text{ ax/ind}}{\bullet h_4 \text{ :- } \Delta_7, F_2, F_5 \oplus F_6} \oplus_B$$

$$\frac{h_4 \text{ :- } F_6, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \oplus F_6} \oplus_B \rightarrow \frac{\overline{h_4 \text{ :- } \Delta_1, F_2} \text{ ax/ind}}{\bullet h_4 \text{ :- } \Delta_1, F_2} \text{ height}$$

- Case rule  $\oplus_A$

$$\frac{h_4 \text{ :- } F_5, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \oplus F_6} \oplus_A \rightarrow \frac{\overline{h_4 \text{ :- } \Delta_7, F_2, F_5} \text{ ax/ind}}{\bullet h_4 \text{ :- } \Delta_7, F_2, F_5 \oplus F_6} \oplus_A$$

$$\frac{h_4 \text{ :- } F_5, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \oplus F_6} \oplus_A \rightarrow \frac{\overline{h_4 \text{ :- } \Delta_1, F_2} \text{ ax/ind}}{\bullet h_4 \text{ :- } \Delta_1, F_2} \text{ height}$$

- Case rule  $\perp$

$$\frac{h_4 \text{ :- } \Delta_1, \Delta_5, F_2 \otimes F_3}{\bullet h_4 \text{ :- } \perp, \Delta_1, \Delta_5, F_2 \otimes F_3} \perp \rightarrow \frac{\overline{h_4 \text{ :- } \Delta_5, F_2} \text{ ax/ind}}{\bullet h_4 \text{ :- } \Delta_5, F_2, \perp} \perp$$

$$\frac{h_4 \text{ :- } \Delta_1, \Delta_5, F_2 \otimes F_3}{\bullet h_4 \text{ :- } \perp, \Delta_1, \Delta_5, F_2 \otimes F_3} \perp \rightarrow \frac{\overline{h_4 \text{ :- } \Delta_1, F_2} \text{ ax/ind}}{\bullet h_4 \text{ :- } \Delta_1, F_2} \text{ height}$$

- Case rule  $\top$

$$\frac{}{\bullet h_4 \text{ :- } \top, \Delta_1, \Delta_5, F_2 \otimes F_3} \top \rightarrow \frac{}{\bullet h_4 \text{ :- } \Delta_5, F_2, \top} \top$$

$$\frac{}{\bullet h_4 \text{ :- } \top, \Delta_1, \Delta_5, F_2 \otimes F_3} \top \rightarrow \frac{}{\bullet h_4 \text{ :- } F_2, \Delta_1} \text{ fail}$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{h_3 \text{ :- } F_4, \Delta_6, \Delta_7, F_1 \otimes F_2 \quad h_3 \text{ :- } F_5, \Delta_8, \Delta_9}{\bullet h_3 \text{ :- } (\Delta_6, \Delta_7, F_1 \otimes F_2), (\Delta_8, \Delta_9), F_4 \otimes F_5} \otimes \rightarrow \frac{}{\bullet h_3 \text{ :- } F_1, \Delta_6, \Delta_8, F_4 \otimes F_5} \text{ fail}$$

$$\frac{h_3 \text{ :- } F_4, \Delta_6, \Delta_7 \quad h_3 \text{ :- } F_5, \Delta_8, \Delta_9, F_1 \otimes F_2}{\bullet h_3 \text{ :- } (\Delta_6, \Delta_7), (\Delta_8, \Delta_9, F_1 \otimes F_2), F_4 \otimes F_5} \otimes \rightarrow \frac{}{\bullet h_3 \text{ :- } F_1, \Delta_6, \Delta_8, F_4 \otimes F_5} \text{ fail}$$

$$\frac{h_3 \text{ :- } F_4, \Delta_6, \Delta_7, F_1 \otimes F_2 \quad h_3 \text{ :- } F_5, \Delta_8, \Delta_9}{\bullet h_3 \text{ :- } (\Delta_6, \Delta_7, F_1 \otimes F_2), (\Delta_8, \Delta_9), F_4 \otimes F_5} \otimes \rightarrow \frac{}{\bullet h_3 \text{ :- } F_1, \Delta_6, \Delta_8} \text{ fail}$$

$$\frac{h_3 \text{ :- } F_4, \Delta_6, \Delta_7 \quad h_3 \text{ :- } F_5, \Delta_8, \Delta_9, F_1 \otimes F_2}{\bullet h_3 \text{ :- } (\Delta_6, \Delta_7), (\Delta_8, \Delta_9, F_1 \otimes F_2), F_4 \otimes F_5} \otimes \rightarrow \frac{}{\bullet h_3 \text{ :- } F_1, \Delta_6, \Delta_8} \text{ fail}$$

$$\frac{h_1 \text{ :- } F_2, \Delta_4, \Delta_5 \quad h_1 \text{ :- } F_3, \Delta_6, \Delta_7}{\bullet h_1 \text{ :- } (\Delta_4, \Delta_5), (\Delta_6, \Delta_7), F_2 \otimes F_3} \otimes \rightarrow \frac{}{\bullet h_1 \text{ :- } F_2, \Delta_4, \Delta_6} \text{ fail}$$

## 2.11 Status of $\otimes$ (Right Premise): : Non invertible

- Case rule 1

- Case rule  $\$$

$$\frac{h_4 \text{ :- } F_5, F_6, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \$ F_6} \$ \rightarrow \frac{h_4 \text{ :- } \Delta_1, F_3}{\bullet h_4 \text{ :- } \Delta_1, F_3} \begin{array}{l} \text{ax/ind} \\ \text{height} \end{array}$$

$$\frac{h_4 \text{ :- } F_5, F_6, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \$ F_6} \$ \rightarrow \frac{h_4 \text{ :- } \Delta_7, F_3, F_5, F_6}{\bullet h_4 \text{ :- } \Delta_7, F_3, F_5 \$ F_6} \begin{array}{l} \text{ax/ind} \\ \$ \end{array}$$

- Case rule  $\&$

$$\frac{h_4 \text{ :- } F_5, \Delta_1, \Delta_7, F_2 \otimes F_3 \quad h_4 \text{ :- } F_6, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \& F_6} \& \rightarrow \frac{h_4 \text{ :- } \Delta_1, F_3}{\bullet h_4 \text{ :- } \Delta_1, F_3} \begin{array}{l} \text{ax/ind} \\ \text{height} \end{array}$$

$$\frac{h_4 \text{ :- } F_5, \Delta_1, \Delta_7, F_2 \otimes F_3 \quad h_4 \text{ :- } F_6, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \& F_6} \& \rightarrow \frac{h_4 \text{ :- } \Delta_7, F_3, F_5}{\bullet h_4 \text{ :- } \Delta_7, F_3, F_5 \& F_6} \begin{array}{l} \text{ax/ind} \\ \& \end{array} \frac{h_4 \text{ :- } \Delta_7, F_3, F_6}{\bullet h_4 \text{ :- } \Delta_7, F_3, F_5 \& F_6} \begin{array}{l} \text{ax/ind} \\ \& \end{array}$$

- Case rule  $\oplus_B$

$$\frac{h_4 \text{ :- } F_6, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \oplus_B F_6} \oplus_B \rightarrow \frac{h_4 \text{ :- } \Delta_1, F_3}{\bullet h_4 \text{ :- } \Delta_1, F_3} \begin{array}{l} \text{ax/ind} \\ \text{height} \end{array}$$

$$\frac{h_4 \text{ :- } F_6, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 \text{ :- } (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \oplus_B F_6} \oplus_B \rightarrow \frac{h_4 \text{ :- } \Delta_7, F_3, F_6}{\bullet h_4 \text{ :- } \Delta_7, F_3, F_5 \oplus_B F_6} \begin{array}{l} \text{ax/ind} \\ \oplus_B \end{array}$$

- Case rule  $\oplus_A$

$$\frac{h_4 : \vdash F_5, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 : \vdash (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \oplus F_6} \oplus_A \rightarrow \frac{\overline{h_4 : \vdash \Delta_1, F_3} \text{ ax/ind}}{\bullet h_4 : \vdash \Delta_1, F_3} \text{ height}$$

$$\frac{h_4 : \vdash F_5, \Delta_1, \Delta_7, F_2 \otimes F_3}{\bullet h_4 : \vdash (\Delta_1, \Delta_7, F_2 \otimes F_3), F_5 \oplus F_6} \oplus_A \rightarrow \frac{\overline{h_4 : \vdash \Delta_7, F_3, F_5} \text{ ax/ind}}{\bullet h_4 : \vdash \Delta_7, F_3, F_5 \oplus F_6} \oplus_A$$

- Case rule  $\perp$

$$\frac{h_4 : \vdash \Delta_1, \Delta_5, F_2 \otimes F_3}{\bullet h_4 : \vdash \perp, \Delta_1, \Delta_5, F_2 \otimes F_3} \perp \rightarrow \frac{\overline{h_4 : \vdash \Delta_1, F_3} \text{ ax/ind}}{\bullet h_4 : \vdash \Delta_1, F_3} \text{ height}$$

$$\frac{h_4 : \vdash \Delta_1, \Delta_5, F_2 \otimes F_3}{\bullet h_4 : \vdash \perp, \Delta_1, \Delta_5, F_2 \otimes F_3} \perp \rightarrow \frac{\overline{h_4 : \vdash \Delta_5, F_3} \text{ ax/ind}}{\bullet h_4 : \vdash \Delta_5, F_3, \perp} \perp$$

- Case rule  $\top$

$$\frac{}{\bullet h_4 : \vdash \top, \Delta_1, \Delta_5, F_2 \otimes F_3} \top \rightarrow \frac{}{\bullet h_4 : \vdash F_3, \Delta_1} \text{ fail}$$

$$\frac{}{\bullet h_4 : \vdash \top, \Delta_1, \Delta_5, F_2 \otimes F_3} \top \rightarrow \frac{}{\bullet h_4 : \vdash \Delta_5, F_3, \top} \top$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{h_3 : \vdash F_4, \Delta_6, \Delta_7, F_1 \otimes F_2 \quad h_3 : \vdash F_5, \Delta_8, \Delta_9}{\bullet h_3 : \vdash (\Delta_6, \Delta_7, F_1 \otimes F_2), (\Delta_8, \Delta_9), F_4 \otimes F_5} \otimes \rightarrow \frac{}{\bullet h_3 : \vdash F_2, \Delta_7, \Delta_9} \text{ fail}$$

$$\frac{h_3 : \vdash F_4, \Delta_6, \Delta_7 \quad h_3 : \vdash F_5, \Delta_8, \Delta_9, F_1 \otimes F_2}{\bullet h_3 : \vdash (\Delta_6, \Delta_7), (\Delta_8, \Delta_9, F_1 \otimes F_2), F_4 \otimes F_5} \otimes \rightarrow \frac{}{\bullet h_3 : \vdash F_2, \Delta_7, \Delta_9} \text{ fail}$$

$$\frac{h_3 : \vdash F_4, \Delta_6, \Delta_7, F_1 \otimes F_2 \quad h_3 : \vdash F_5, \Delta_8, \Delta_9}{\bullet h_3 : \vdash (\Delta_6, \Delta_7, F_1 \otimes F_2), (\Delta_8, \Delta_9), F_4 \otimes F_5} \otimes \rightarrow \frac{}{\bullet h_3 : \vdash F_2, \Delta_7, \Delta_9, F_4 \otimes F_5} \text{ fail}$$

$$\frac{h_3 : \vdash F_4, \Delta_6, \Delta_7 \quad h_3 : \vdash F_5, \Delta_8, \Delta_9, F_1 \otimes F_2}{\bullet h_3 : \vdash (\Delta_6, \Delta_7), (\Delta_8, \Delta_9, F_1 \otimes F_2), F_4 \otimes F_5} \otimes \rightarrow \frac{}{\bullet h_3 : \vdash F_2, \Delta_7, \Delta_9, F_4 \otimes F_5} \text{ fail}$$

$$\frac{h_1 : \vdash F_2, \Delta_4, \Delta_5 \quad h_1 : \vdash F_3, \Delta_6, \Delta_7}{\bullet h_1 : \vdash (\Delta_4, \Delta_5), (\Delta_6, \Delta_7), F_2 \otimes F_3} \otimes \rightarrow \frac{}{\bullet h_1 : \vdash F_3, \Delta_5, \Delta_7} \text{ fail}$$

### 3 Identity-Expansion

$$\begin{array}{c}
\frac{}{- : \vdash \mathbf{0}, \top} \top \\
\\
\frac{\frac{}{- : \vdash \mathbf{1}}{\quad} \mathbf{1}}{- : \vdash \mathbf{1}, \perp} \perp \\
\\
\frac{\frac{\frac{}{- : \vdash \mathbf{F}_0, \text{dual}(\mathbf{F}_0)}{\quad} \text{IH}}{- : \vdash \text{dual}(\mathbf{F}_0), \mathbf{F}_0 \oplus \mathbf{F}_1} \oplus_A \quad \frac{\frac{}{- : \vdash \mathbf{F}_1, \text{dual}(\mathbf{F}_1)}{\quad} \text{IH}}{- : \vdash \text{dual}(\mathbf{F}_1), \mathbf{F}_0 \oplus \mathbf{F}_1} \oplus_B}{- : \vdash \text{dual}(\mathbf{F}_0) \& \text{dual}(\mathbf{F}_1), \mathbf{F}_0 \oplus \mathbf{F}_1} \& \\
\\
\frac{\frac{\frac{}{- : \vdash \mathbf{F}_0, \text{dual}(\mathbf{F}_0)}{\quad} \text{IH}}{- : \vdash \text{dual}(\mathbf{F}_0), \text{dual}(\mathbf{F}_1), \mathbf{F}_0 \otimes \mathbf{F}_1} \otimes \quad \frac{\frac{}{- : \vdash \mathbf{F}_1, \text{dual}(\mathbf{F}_1)}{\quad} \text{IH}}{- : \vdash \text{dual}(\mathbf{F}_0) \$ \text{dual}(\mathbf{F}_1), \mathbf{F}_0 \otimes \mathbf{F}_1} \$}{- : \vdash \text{dual}(\mathbf{F}_0), \text{dual}(\mathbf{F}_1), \mathbf{F}_0 \otimes \mathbf{F}_1} \otimes \\
\\
\frac{\frac{\frac{}{- : \vdash \mathbf{F}_0, \text{dual}(\mathbf{F}_0)}{\quad} \text{IH}}{- : \vdash \mathbf{F}_0, \text{dual}(\mathbf{F}_0) \oplus \text{dual}(\mathbf{F}_1)} \oplus_A \quad \frac{\frac{}{- : \vdash \mathbf{F}_1, \text{dual}(\mathbf{F}_1)}{\quad} \text{IH}}{- : \vdash \mathbf{F}_1, \text{dual}(\mathbf{F}_0) \oplus \text{dual}(\mathbf{F}_1)} \oplus_B}{- : \vdash \mathbf{F}_0 \& \mathbf{F}_1, \text{dual}(\mathbf{F}_0) \oplus \text{dual}(\mathbf{F}_1)} \& \\
\\
\frac{\frac{\frac{}{- : \vdash \mathbf{F}_0, \text{dual}(\mathbf{F}_0)}{\quad} \text{IH}}{- : \vdash \mathbf{F}_0, \mathbf{F}_1, \text{dual}(\mathbf{F}_0) \otimes \text{dual}(\mathbf{F}_1)} \otimes \quad \frac{\frac{}{- : \vdash \mathbf{F}_1, \text{dual}(\mathbf{F}_1)}{\quad} \text{IH}}{- : \vdash \mathbf{F}_0 \$ \mathbf{F}_1, \text{dual}(\mathbf{F}_0) \otimes \text{dual}(\mathbf{F}_1)} \$}{- : \vdash \mathbf{F}_0, \mathbf{F}_1, \text{dual}(\mathbf{F}_0) \otimes \text{dual}(\mathbf{F}_1)} \otimes \\
\\
\frac{}{- : \vdash \mathbf{1}}{\quad} \mathbf{1} \\
\frac{}{- : \vdash \mathbf{1}, \perp} \perp \\
\\
\frac{}{- : \vdash \mathbf{0}, \top} \top
\end{array}$$

## 4 Cut-Elimination

### 4.1 Status of 1: OK

- Case rule 1
- Case rule \$

$$\frac{\frac{\frac{}{\bullet h_1 : \vdash \mathbf{1}, *}}{\bullet h_1 : \vdash \mathbf{1}, *} \mathbf{1} \quad \frac{h_2 : \vdash \perp, F_3, F_4, \Delta_5}{\bullet h_2 : \vdash \mathit{dual}(\mathbf{1}), \Delta_5, F_3 \$ F_4} \$}{- : \vdash *, \Delta_5, F_3 \$ F_4} \text{Cut}}{\rightarrow} \frac{\frac{\frac{}{\bullet h_1 : \vdash \mathbf{1}}{\bullet h_1 : \vdash \mathbf{1}} \text{ax} \quad \frac{h_2 : \vdash \Delta_5, F_3, F_4, \perp}{h_2 : \vdash \Delta_5, F_3, F_4, \perp} \text{ax}}{- : \vdash \Delta_5, F_3, F_4} \text{hCut}}{- : \vdash \Delta_5, F_3 \$ F_4} \$}{- : \vdash \Delta_5, F_3 \$ F_4} \$$$

- Case rule &

$$\frac{\frac{\frac{}{\bullet h_1 : \vdash \mathbf{1}, *}}{\bullet h_1 : \vdash \mathbf{1}, *} \mathbf{1} \quad \frac{h_2 : \vdash \perp, F_3, \Delta_5 \quad h_2 : \vdash \perp, F_4, \Delta_5}{\bullet h_2 : \vdash \mathit{dual}(\mathbf{1}), \Delta_5, F_3 \& F_4} \&}{- : \vdash *, \Delta_5, F_3 \& F_4} \text{Cut}}{\rightarrow} \frac{\frac{\frac{}{\bullet h_1 : \vdash \mathbf{1}}{\bullet h_1 : \vdash \mathbf{1}} \text{ax} \quad \frac{h_2 : \vdash \Delta_5, F_3, \perp}{h_2 : \vdash \Delta_5, F_3, \perp} \text{ax}}{- : \vdash \Delta_5, F_3} \text{hCut} \quad \frac{\frac{}{\bullet h_1 : \vdash \mathbf{1}}{\bullet h_1 : \vdash \mathbf{1}} \text{ax} \quad \frac{h_2 : \vdash \Delta_5, F_4, \perp}{h_2 : \vdash \Delta_5, F_4, \perp} \text{ax}}{- : \vdash \Delta_5, F_4} \text{hCut}}{- : \vdash \Delta_5, F_3 \& F_4} \&}{- : \vdash \Delta_5, F_3 \& F_4} \&$$

- Case rule  $\oplus_B$

$$\frac{\frac{\frac{}{\bullet h_1 : \vdash \mathbf{1}, *}}{\bullet h_1 : \vdash \mathbf{1}, *} \mathbf{1} \quad \frac{h_2 : \vdash \perp, F_4, \Delta_5}{\bullet h_2 : \vdash \mathit{dual}(\mathbf{1}), \Delta_5, F_3 \oplus F_4} \oplus_B}{- : \vdash *, \Delta_5, F_3 \oplus F_4} \text{Cut}}{\rightarrow} \frac{\frac{\frac{}{\bullet h_1 : \vdash \mathbf{1}}{\bullet h_1 : \vdash \mathbf{1}} \text{ax} \quad \frac{h_2 : \vdash \Delta_5, F_4, \perp}{h_2 : \vdash \Delta_5, F_4, \perp} \text{ax}}{- : \vdash \Delta_5, F_4} \text{hCut}}{- : \vdash \Delta_5, F_3 \oplus F_4} \oplus_B}{- : \vdash \Delta_5, F_3 \oplus F_4} \oplus_B$$

- Case rule  $\oplus_A$

$$\frac{\frac{\frac{}{\bullet h_1 : \vdash \mathbf{1}, *}}{\bullet h_1 : \vdash \mathbf{1}, *} \mathbf{1} \quad \frac{h_2 : \vdash \perp, F_3, \Delta_5}{\bullet h_2 : \vdash \mathit{dual}(\mathbf{1}), \Delta_5, F_3 \oplus F_4} \oplus_A}{- : \vdash *, \Delta_5, F_3 \oplus F_4} \text{Cut}}{\rightarrow} \frac{\frac{\frac{}{\bullet h_1 : \vdash \mathbf{1}}{\bullet h_1 : \vdash \mathbf{1}} \text{ax} \quad \frac{h_2 : \vdash \Delta_5, F_3, \perp}{h_2 : \vdash \Delta_5, F_3, \perp} \text{ax}}{- : \vdash \Delta_5, F_3} \text{hCut}}{- : \vdash \Delta_5, F_3 \oplus F_4} \oplus_A}{- : \vdash \Delta_5, F_3 \oplus F_4} \oplus_A$$

- Case rule  $\perp$

$$\frac{\frac{\frac{}{\bullet h_1 : \vdash \mathbf{1}, *}}{\bullet h_1 : \vdash \mathbf{1}, *} \mathbf{1} \quad \frac{h_2 : \vdash \Delta_3}{\bullet h_2 : \vdash \mathit{dual}(\mathbf{1}), \Delta_3} \perp}{- : \vdash *, \Delta_3} \text{Cut}}{\rightarrow} \frac{}{- : \vdash \Delta_3} \text{ax}$$

- Case rule  $\top$

$$\frac{\frac{\bullet h_1 : \vdash \mathbf{1}, *}{\bullet h_2 : \vdash \mathbf{1}, *}}{\vdash : \vdash *, \top, \Delta_3} \top \quad \frac{\bullet h_2 : \vdash \mathbf{1}, *}{\vdash : \vdash *, \top, \Delta_3} \top}{\vdash : \vdash \Delta_3, \top} \text{Cut}$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{\frac{\bullet h_1 : \vdash \mathbf{1}, *}{\bullet h_2 : \vdash \mathbf{1}, *}}{\vdash : \vdash *, \Delta_3, \Delta_6, F_4 \otimes F_5} \top \quad \frac{h_2 : \vdash \perp, F_4, \Delta_6 \quad h_2 : \vdash F_5, \Delta_3}{\bullet h_2 : \vdash \mathbf{1}, *}}{\vdash : \vdash *, \Delta_3, \Delta_6, F_4 \otimes F_5} \otimes}{\vdash : \vdash \Delta_3, \Delta_6, F_4 \otimes F_5} \text{Cut}$$

$$\frac{\frac{\bullet h_1 : \vdash \mathbf{1}}{\vdash : \vdash \Delta_6, F_4} \text{ax} \quad \frac{h_2 : \vdash \Delta_6, F_4, \perp}{\vdash : \vdash \Delta_3, F_5} \text{ax}}{\vdash : \vdash \Delta_3, \Delta_6, F_4 \otimes F_5} \text{hCut} \quad \frac{\vdash : \vdash \Delta_3, F_5}{\vdash : \vdash \Delta_3, \Delta_6, F_4 \otimes F_5} \otimes}{\vdash : \vdash \Delta_3, \Delta_6, F_4 \otimes F_5} \otimes$$

$$\frac{\frac{\bullet h_1 : \vdash \mathbf{1}, *}{\bullet h_2 : \vdash \mathbf{1}, *}}{\vdash : \vdash *, \Delta_3, \Delta_6, F_4 \otimes F_5} \top \quad \frac{h_2 : \vdash F_4, \Delta_3 \quad h_2 : \vdash \perp, F_5, \Delta_6}{\bullet h_2 : \vdash \mathbf{1}, *}}{\vdash : \vdash *, \Delta_3, \Delta_6, F_4 \otimes F_5} \otimes}{\vdash : \vdash \Delta_3, \Delta_6, F_4 \otimes F_5} \text{Cut}$$

$$\frac{\frac{\vdash : \vdash \Delta_3, F_4}{\vdash : \vdash \Delta_3, F_4} \text{ax} \quad \frac{\bullet h_1 : \vdash \mathbf{1}}{\vdash : \vdash \Delta_6, F_5} \text{ax} \quad \frac{h_2 : \vdash \Delta_6, F_5, \perp}{\vdash : \vdash \Delta_6, F_5} \text{ax}}{\vdash : \vdash \Delta_3, \Delta_6, F_4 \otimes F_5} \text{hCut} \quad \frac{\vdash : \vdash \Delta_6, F_5}{\vdash : \vdash \Delta_3, \Delta_6, F_4 \otimes F_5} \otimes}{\vdash : \vdash \Delta_3, \Delta_6, F_4 \otimes F_5} \otimes$$

## 4.2 Status of \$: OK

- Case rule  $\mathbf{1}$

- Case rule  $\$$

$$\frac{\frac{h_1 : \vdash F_5, F_6, \Delta_2}{\bullet h_1 : \vdash F_5 \$ F_6, \Delta_2} \$ \quad \frac{h_7 : \vdash F_8, F_9, \Delta_{10}, \mathit{dual}(F_5) \otimes \mathit{dual}(F_6)}{\bullet h_7 : \vdash \mathit{dual}(F_5 \$ F_6), \Delta_{10}, F_8 \$ F_9} \$}{\vdash : \vdash \Delta_2, \Delta_{10}, F_8 \$ F_9} \text{Cut}}{\vdash : \vdash \Delta_2, \Delta_{10}, F_8 \$ F_9} \text{Cut}$$

$$\frac{\frac{\bullet h_1 : \vdash \Delta_2, F_5 \$ F_6}{\vdash : \vdash \Delta_{10}, \Delta_2, F_8, F_9} \text{ax} \quad \frac{h_7 : \vdash \Delta_{10}, F_8, F_9, \mathit{dual}(F_5) \otimes \mathit{dual}(F_6)}{\vdash : \vdash \Delta_{10}, \Delta_2, F_8 \$ F_9} \text{ax}}{\vdash : \vdash \Delta_{10}, \Delta_2, F_8 \$ F_9} \text{hCut} \quad \frac{\vdash : \vdash \Delta_{10}, \Delta_2, F_8, F_9}{\vdash : \vdash \Delta_{10}, \Delta_2, F_8 \$ F_9} \$}{\vdash : \vdash \Delta_{10}, \Delta_2, F_8 \$ F_9} \$$$

$$\frac{\frac{h_2 : \vdash F_6, F_3, F_4, \Delta_5}{\bullet h_2 : \vdash F_6, \Delta_5, F_3 \$ F_4} \$ \quad \frac{h_7 : \vdash F_8, F_9, \Delta_{10}, \mathit{dual}(F_6)}{\bullet h_7 : \vdash \mathit{dual}(F_6), \Delta_{10}, F_8 \$ F_9} \$}{\vdash : \vdash (\Delta_5, F_3 \$ F_4), \Delta_{10}, F_8 \$ F_9} \text{Cut}}{\vdash : \vdash (\Delta_5, F_3 \$ F_4), \Delta_{10}, F_8 \$ F_9} \text{Cut}$$

$$\frac{\frac{h_2 : \vdash \Delta_5, F_3, F_4, F_6}{\vdash : \vdash \Delta_{10}, \Delta_5, F_3, F_4, F_8 \$ F_9} \text{ax} \quad \frac{\bullet h_7 : \vdash \Delta_{10}, \mathit{dual}(F_6), F_8 \$ F_9}{\vdash : \vdash \Delta_{10}, \Delta_5, F_3 \$ F_4, F_8 \$ F_9} \text{ax}}{\vdash : \vdash \Delta_{10}, \Delta_5, F_3, F_4, F_8 \$ F_9} \text{hCut} \quad \frac{\vdash : \vdash \Delta_{10}, \Delta_5, F_3, F_4, F_8 \$ F_9}{\vdash : \vdash \Delta_{10}, \Delta_5, F_3 \$ F_4, F_8 \$ F_9} \$}{\vdash : \vdash \Delta_{10}, \Delta_5, F_3 \$ F_4, F_8 \$ F_9} \$$$

- Case rule  $\&$

$$\frac{\frac{h_1 : \vdash F_5, F_6, \Delta_2}{\bullet h_1 : \vdash F_5 \$ F_6, \Delta_2} \$ \quad \frac{h_7 : \vdash F_8, \Delta_{10}, \mathit{dual}(F_5) \otimes \mathit{dual}(F_6) \quad h_7 : \vdash F_9, \Delta_{10}, \mathit{dual}(F_5) \otimes \mathit{dual}(F_6)}{\bullet h_7 : \vdash \mathit{dual}(F_5 \$ F_6), \Delta_{10}, F_8 \& F_9} \&}{\vdash : \vdash \Delta_2, \Delta_{10}, F_8 \& F_9} \text{Cut}}{\vdash : \vdash \Delta_2, \Delta_{10}, F_8 \& F_9} \text{Cut}$$

$$\frac{\frac{\bullet h_1 : \vdash \Delta_2, F_5 \$ F_6}{\vdash : \vdash \Delta_{10}, \Delta_2, F_8} \text{ax} \quad \frac{h_7 : \vdash \Delta_{10}, F_8, \mathit{dual}(F_5) \otimes \mathit{dual}(F_6)}{\vdash : \vdash \Delta_{10}, \Delta_2, F_8} \text{ax}}{\vdash : \vdash \Delta_{10}, \Delta_2, F_8} \text{hCut} \quad \frac{\bullet h_1 : \vdash \Delta_2, F_5 \$ F_6}{\vdash : \vdash \Delta_{10}, \Delta_2, F_9} \text{ax} \quad \frac{h_7 : \vdash \Delta_{10}, F_9, \mathit{dual}(F_5) \otimes \mathit{dual}(F_6)}{\vdash : \vdash \Delta_{10}, \Delta_2, F_9} \text{ax}}{\vdash : \vdash \Delta_{10}, \Delta_2, F_9} \text{hCut} \quad \frac{\vdash : \vdash \Delta_{10}, \Delta_2, F_8}{\vdash : \vdash \Delta_{10}, \Delta_2, F_8 \& F_9} \&}{\vdash : \vdash \Delta_{10}, \Delta_2, F_8 \& F_9} \&$$



$$\begin{array}{c}
\frac{h_2 \vdash F_6, F_3, F_4, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \& F_4} \text{\$} \quad \frac{h_7 \vdash F_8, \Delta_{10}, \text{dual}(F_6) \quad h_7 \vdash F_9, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_6), \Delta_{10}, F_8 \& F_9} \text{\&} \\
\hline
- \vdash (\Delta_5, F_3 \& F_4), \Delta_{10}, F_8 \& F_9 \\
\rightarrow \\
\frac{\frac{h_2 \vdash \Delta_5, F_3, F_4, F_6}{- \vdash \Delta_{10}, \Delta_5, F_3, F_4, F_8 \& F_9} \text{ax} \quad \frac{\bullet h_7 \vdash \Delta_{10}, \text{dual}(F_6), F_8 \& F_9}{- \vdash \Delta_{10}, \Delta_5, F_3 \& F_4, F_8 \& F_9} \text{ax}}{- \vdash \Delta_{10}, \Delta_5, F_3 \& F_4, F_8 \& F_9} \text{hCut} \text{\$}
\end{array}$$

- Case rule  $\oplus_B$

$$\begin{array}{c}
\frac{h_1 \vdash F_5, F_6, \Delta_2}{\bullet h_1 \vdash F_5 \& F_6, \Delta_2} \text{\$} \quad \frac{h_7 \vdash F_9, \Delta_{10}, \text{dual}(F_5) \otimes \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_5 \& F_6), \Delta_{10}, F_8 \oplus F_9} \text{\oplus}_B \\
\hline
- \vdash \Delta_2, \Delta_{10}, F_8 \oplus F_9 \\
\rightarrow \\
\frac{\frac{\bullet h_1 \vdash \Delta_2, F_5 \& F_6}{- \vdash \Delta_{10}, \Delta_2, F_9} \text{ax} \quad \frac{h_7 \vdash \Delta_{10}, F_9, \text{dual}(F_5) \otimes \text{dual}(F_6)}{- \vdash \Delta_{10}, \Delta_2, F_8 \oplus F_9} \text{ax}}{- \vdash \Delta_{10}, \Delta_2, F_8 \oplus F_9} \text{hCut} \text{\oplus}_B
\end{array}$$

$$\begin{array}{c}
\frac{h_2 \vdash F_6, F_3, F_4, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \& F_4} \text{\$} \quad \frac{h_7 \vdash F_9, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_6), \Delta_{10}, F_8 \oplus F_9} \text{\oplus}_B \\
\hline
- \vdash (\Delta_5, F_3 \& F_4), \Delta_{10}, F_8 \oplus F_9 \\
\rightarrow \\
\frac{\frac{h_2 \vdash \Delta_5, F_3, F_4, F_6}{- \vdash \Delta_{10}, \Delta_5, F_3, F_4, F_8 \oplus F_9} \text{ax} \quad \frac{\bullet h_7 \vdash \Delta_{10}, \text{dual}(F_6), F_8 \oplus F_9}{- \vdash \Delta_{10}, \Delta_5, F_3 \& F_4, F_8 \oplus F_9} \text{ax}}{- \vdash \Delta_{10}, \Delta_5, F_3 \& F_4, F_8 \oplus F_9} \text{hCut} \text{\$}
\end{array}$$

- Case rule  $\oplus_A$

$$\begin{array}{c}
\frac{h_1 \vdash F_5, F_6, \Delta_2}{\bullet h_1 \vdash F_5 \& F_6, \Delta_2} \text{\$} \quad \frac{h_7 \vdash F_8, \Delta_{10}, \text{dual}(F_5) \otimes \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_5 \& F_6), \Delta_{10}, F_8 \oplus F_9} \text{\oplus}_A \\
\hline
- \vdash \Delta_2, \Delta_{10}, F_8 \oplus F_9 \\
\rightarrow \\
\frac{\frac{\bullet h_1 \vdash \Delta_2, F_5 \& F_6}{- \vdash \Delta_{10}, \Delta_2, F_8} \text{ax} \quad \frac{h_7 \vdash \Delta_{10}, F_8, \text{dual}(F_5) \otimes \text{dual}(F_6)}{- \vdash \Delta_{10}, \Delta_2, F_8 \oplus F_9} \text{ax}}{- \vdash \Delta_{10}, \Delta_2, F_8 \oplus F_9} \text{hCut} \text{\oplus}_A
\end{array}$$

$$\begin{array}{c}
\frac{h_2 \vdash F_6, F_3, F_4, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \& F_4} \text{\$} \quad \frac{h_7 \vdash F_8, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_6), \Delta_{10}, F_8 \oplus F_9} \text{\oplus}_A \\
\hline
- \vdash (\Delta_5, F_3 \& F_4), \Delta_{10}, F_8 \oplus F_9 \\
\rightarrow \\
\frac{\frac{h_2 \vdash \Delta_5, F_3, F_4, F_6}{- \vdash \Delta_{10}, \Delta_5, F_3, F_4, F_8 \oplus F_9} \text{ax} \quad \frac{\bullet h_7 \vdash \Delta_{10}, \text{dual}(F_6), F_8 \oplus F_9}{- \vdash \Delta_{10}, \Delta_5, F_3 \& F_4, F_8 \oplus F_9} \text{ax}}{- \vdash \Delta_{10}, \Delta_5, F_3 \& F_4, F_8 \oplus F_9} \text{hCut} \text{\$}
\end{array}$$

- Case rule  $\perp$

$$\begin{array}{c}
\frac{h_1 \vdash F_5, F_6, \Delta_2}{\bullet h_1 \vdash F_5 \& F_6, \Delta_2} \text{\$} \quad \frac{h_7 \vdash \Delta_8, \text{dual}(F_5) \otimes \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_5 \& F_6), \perp, \Delta_8} \text{\perp} \\
\hline
- \vdash \Delta_2, \perp, \Delta_8 \\
\rightarrow \\
\frac{\frac{\bullet h_1 \vdash \Delta_2, F_5 \& F_6}{- \vdash \Delta_2, \Delta_8} \text{ax} \quad \frac{h_7 \vdash \Delta_8, \text{dual}(F_5) \otimes \text{dual}(F_6)}{- \vdash \Delta_2, \Delta_8, \perp} \text{ax}}{- \vdash \Delta_2, \Delta_8, \perp} \text{hCut} \text{\perp}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 \vdash F_6, F_3, F_4, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \& F_4} \text{\$} \quad \frac{h_7 \vdash \Delta_8, \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_6), \perp, \Delta_8} \text{\perp} \\
\hline
- \vdash (\Delta_5, F_3 \& F_4), \perp, \Delta_8 \\
\rightarrow \\
\frac{\frac{h_2 \vdash \Delta_5, F_3, F_4, F_6}{- \vdash \Delta_5, \Delta_8, F_3, F_4, \perp} \text{ax} \quad \frac{\bullet h_7 \vdash \Delta_8, \perp, \text{dual}(F_6)}{- \vdash \Delta_5, \Delta_8, \perp, F_3 \& F_4} \text{ax}}{- \vdash \Delta_5, \Delta_8, \perp, F_3 \& F_4} \text{hCut} \text{\$}
\end{array}$$

- Case rule  $\top$

$$\frac{\frac{h_1 \vdash F_5, F_6, \Delta_2}{\bullet h_1 \vdash F_5 \S F_6, \Delta_2} \S \frac{\top}{\bullet h_7 \vdash \text{dual}(F_5 \S F_6), \top, \Delta_8} \top}{- \vdash \Delta_2, \top, \Delta_8} \text{Cut}$$

$$\xrightarrow{\quad}$$

$$\frac{}{- \vdash \Delta_2, \Delta_8, \top} \top$$
  

$$\frac{\frac{h_2 \vdash F_6, F_3, F_4, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \S F_4} \S \frac{\top}{\bullet h_7 \vdash \text{dual}(F_6), \top, \Delta_8} \top}{- \vdash (\Delta_5, F_3 \S F_4), \top, \Delta_8} \text{Cut}$$

$$\xrightarrow{\quad}$$

$$\frac{}{- \vdash \Delta_5, \Delta_8, \top, F_3 \S F_4} \top$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{\frac{h_1 \vdash F_5, F_6, \Delta_2}{\bullet h_1 \vdash F_5 \S F_6, \Delta_2} \S \frac{\frac{h_7 \vdash F_9, \Delta_{11}, \text{dual}(F_5) \otimes \text{dual}(F_6) \quad h_7 \vdash F_{10}, \Delta_8}{\bullet h_7 \vdash \text{dual}(F_5 \S F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \otimes}{- \vdash \Delta_2, \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}}{\xrightarrow{\quad}}$$

$$\frac{\frac{\frac{h_1 \vdash F_5, F_6, \Delta_2}{\bullet h_1 \vdash \Delta_2, F_5 \S F_6} \text{ax} \quad \frac{h_7 \vdash \Delta_{11}, F_9, \text{dual}(F_5) \otimes \text{dual}(F_6)}{\bullet h_7 \vdash \Delta_{11}, \Delta_2, F_9} \text{ax}}{- \vdash \Delta_{11}, \Delta_2, F_9} \text{hCut} \quad \frac{}{- \vdash \Delta_8, F_{10}} \text{ax}}{- \vdash \Delta_{11}, \Delta_2, \Delta_8, F_9 \otimes F_{10}} \otimes$$
  

$$\frac{\frac{h_1 \vdash F_5, F_6, \Delta_2}{\bullet h_1 \vdash F_5 \S F_6, \Delta_2} \S \frac{\frac{h_7 \vdash F_9, \Delta_8 \quad h_7 \vdash F_{10}, \Delta_{11}, \text{dual}(F_5) \otimes \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_5 \S F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \otimes}{- \vdash \Delta_2, \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}}{\xrightarrow{\quad}}$$

$$\frac{\frac{\frac{h_1 \vdash F_5, F_6, \Delta_2}{\bullet h_1 \vdash \Delta_2, F_5 \S F_6} \text{ax} \quad \frac{h_7 \vdash \Delta_{11}, F_{10}, \text{dual}(F_5) \otimes \text{dual}(F_6)}{\bullet h_7 \vdash \Delta_{11}, \Delta_2, F_{10}} \text{ax}}{- \vdash \Delta_8, F_9} \text{ax} \quad \frac{}{- \vdash \Delta_{11}, \Delta_2, F_{10}} \text{hCut}}{- \vdash \Delta_{11}, \Delta_2, \Delta_8, F_9 \otimes F_{10}} \otimes$$
  

$$\frac{\frac{h_1 \vdash F_5, F_6, \Delta_2}{\bullet h_1 \vdash F_5 \S F_6, \Delta_2} \S \frac{\frac{h_7 \vdash \Delta_8, \text{dual}(F_5) \quad h_7 \vdash \Delta_9, \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_5 \S F_6), \Delta_8, \Delta_9} \otimes}{- \vdash \Delta_2, \Delta_8, \Delta_9} \text{Cut}}{\xrightarrow{\quad}}$$

$$\frac{\frac{\frac{h_1 \vdash F_5, F_6, \Delta_2}{\bullet h_1 \vdash \Delta_2, F_5, F_6} \text{ax} \quad \frac{h_7 \vdash \Delta_9, \text{dual}(F_6)}{\bullet h_7 \vdash \Delta_9, \text{dual}(F_6)} \text{ax}}{- \vdash \Delta_2, \Delta_9, F_5} \text{sCut} \quad \frac{}{- \vdash \Delta_8, \text{dual}(F_5)} \text{ax}}{- \vdash \Delta_2, \Delta_8, \Delta_9} \text{sCut}$$
  

$$\frac{\frac{h_2 \vdash F_6, F_3, F_4, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \S F_4} \S \frac{\frac{h_7 \vdash F_9, \Delta_{11}, \text{dual}(F_6) \quad h_7 \vdash F_{10}, \Delta_8}{\bullet h_7 \vdash \text{dual}(F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \otimes}{- \vdash (\Delta_5, F_3 \S F_4), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}}{\xrightarrow{\quad}}$$

$$\frac{\frac{\frac{h_2 \vdash \Delta_5, F_3, F_4, F_6}{\bullet h_2 \vdash \Delta_{11}, \Delta_8, \text{dual}(F_6), F_9 \otimes F_{10}} \text{ax}}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_3, F_4, F_9 \otimes F_{10}} \text{hCut} \quad \frac{}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_3 \S F_4, F_9 \otimes F_{10}} \S}}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_3, F_4, F_9 \otimes F_{10}} \S$$
  

$$\frac{\frac{h_2 \vdash F_6, F_3, F_4, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \S F_4} \S \frac{\frac{h_7 \vdash F_9, \Delta_8 \quad h_7 \vdash F_{10}, \Delta_{11}, \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \otimes}{- \vdash (\Delta_5, F_3 \S F_4), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}}{\xrightarrow{\quad}}$$

$$\frac{\frac{\frac{h_2 \vdash \Delta_5, F_3, F_4, F_6}{\bullet h_2 \vdash \Delta_{11}, \Delta_8, \text{dual}(F_6), F_9 \otimes F_{10}} \text{ax}}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_3, F_4, F_9 \otimes F_{10}} \text{hCut} \quad \frac{}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_3 \S F_4, F_9 \otimes F_{10}} \S}}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_3 \S F_4, F_9 \otimes F_{10}} \S$$

### 4.3 Status of &: OK

- Case rule 1
- Case rule \$

$$\begin{array}{c}
\frac{\frac{h_1 \text{ : } \vdash F_5, \Delta_2 \quad h_1 \text{ : } \vdash F_6, \Delta_2}{\bullet h_1 \text{ : } \vdash F_5 \& F_6, \Delta_2} \& \quad \frac{h_7 \text{ : } \vdash F_8, F_9, \Delta_{10}, \text{dual}(F_5) \oplus \text{dual}(F_6)}{\bullet h_7 \text{ : } \vdash \text{dual}(F_5 \& F_6), \Delta_{10}, F_8 \& F_9}}{- \text{ : } \vdash \Delta_2, \Delta_{10}, F_8 \& F_9} \text{Cut}}{\rightarrow} \\
\frac{\frac{\bullet h_1 \text{ : } \vdash \Delta_2, F_5 \& F_6}{- \text{ : } \vdash \Delta_{10}, \Delta_2, F_8, F_9} \text{ax} \quad \frac{h_7 \text{ : } \vdash \Delta_{10}, F_8, F_9, \text{dual}(F_5) \oplus \text{dual}(F_6)}{- \text{ : } \vdash \Delta_{10}, \Delta_2, F_8 \& F_9} \text{ax}}{- \text{ : } \vdash \Delta_{10}, \Delta_2, F_8 \& F_9} \text{hCut}}{\$} \\
\frac{\frac{h_2 \text{ : } \vdash F_6, F_3, \Delta_5 \quad h_2 \text{ : } \vdash F_6, F_4, \Delta_5}{\bullet h_2 \text{ : } \vdash F_6, \Delta_5, F_3 \& F_4} \& \quad \frac{h_7 \text{ : } \vdash F_8, F_9, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \text{ : } \vdash \text{dual}(F_6), \Delta_{10}, F_8 \& F_9} \&}{- \text{ : } \vdash (\Delta_5, F_3 \& F_4), \Delta_{10}, F_8 \& F_9} \text{Cut}}{\rightarrow} \\
\frac{\frac{\bullet h_2 \text{ : } \vdash \Delta_5, F_6, F_3 \& F_4}{- \text{ : } \vdash \Delta_{10}, \Delta_5, F_8, F_9, F_3 \& F_4} \text{ax} \quad \frac{h_7 \text{ : } \vdash \Delta_{10}, F_8, F_9, \text{dual}(F_6)}{- \text{ : } \vdash \Delta_{10}, \Delta_5, F_8 \& F_9, F_3 \& F_4} \text{ax}}{- \text{ : } \vdash \Delta_{10}, \Delta_5, F_8 \& F_9, F_3 \& F_4} \text{hCut}}{\$}
\end{array}$$

- Case rule &

$$\begin{array}{c}
\frac{\frac{\frac{h_1 \text{ : } \vdash F_5, \Delta_2 \quad h_1 \text{ : } \vdash F_6, \Delta_2}{\bullet h_1 \text{ : } \vdash F_5 \& F_6, \Delta_2} \& \quad \frac{h_7 \text{ : } \vdash F_8, \Delta_{10}, \text{dual}(F_5) \oplus \text{dual}(F_6)}{\bullet h_7 \text{ : } \vdash \text{dual}(F_5 \& F_6), \Delta_{10}, F_8 \& F_9} \& \quad \frac{h_7 \text{ : } \vdash F_9, \Delta_{10}, \text{dual}(F_5) \oplus \text{dual}(F_6)}{\bullet h_7 \text{ : } \vdash \text{dual}(F_5 \& F_6), \Delta_{10}, F_8 \& F_9} \&}{- \text{ : } \vdash \Delta_2, \Delta_{10}, F_8 \& F_9} \text{Cut}}{\rightarrow} \\
\frac{\frac{\bullet h_1 \text{ : } \vdash \Delta_2, F_5 \& F_6}{- \text{ : } \vdash \Delta_{10}, \Delta_2, F_8} \text{ax} \quad \frac{h_7 \text{ : } \vdash \Delta_{10}, F_8, \text{dual}(F_5) \oplus \text{dual}(F_6)}{- \text{ : } \vdash \Delta_{10}, \Delta_2, F_8 \& F_9} \text{ax}}{- \text{ : } \vdash \Delta_{10}, \Delta_2, F_8 \& F_9} \text{hCut}}{\&} \\
\frac{\frac{h_2 \text{ : } \vdash F_6, F_3, \Delta_5 \quad h_2 \text{ : } \vdash F_6, F_4, \Delta_5}{\bullet h_2 \text{ : } \vdash F_6, \Delta_5, F_3 \& F_4} \& \quad \frac{h_7 \text{ : } \vdash F_8, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \text{ : } \vdash \text{dual}(F_6), \Delta_{10}, F_8 \& F_9} \& \quad \frac{h_7 \text{ : } \vdash F_9, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \text{ : } \vdash \text{dual}(F_6), \Delta_{10}, F_8 \& F_9} \&}{- \text{ : } \vdash (\Delta_5, F_3 \& F_4), \Delta_{10}, F_8 \& F_9} \text{Cut}}{\rightarrow} \\
\frac{\frac{h_2 \text{ : } \vdash \Delta_5, F_3, F_6}{- \text{ : } \vdash \Delta_{10}, \Delta_5, F_3, F_8 \& F_9} \text{ax} \quad \frac{\bullet h_7 \text{ : } \vdash \Delta_{10}, \text{dual}(F_6), F_8 \& F_9}{- \text{ : } \vdash \Delta_{10}, \Delta_5, F_3 \& F_4, F_8 \& F_9} \text{ax}}{- \text{ : } \vdash \Delta_{10}, \Delta_5, F_3 \& F_4, F_8 \& F_9} \text{hCut}}{\&} \\
\frac{\frac{h_2 \text{ : } \vdash \Delta_5, F_4, F_6}{- \text{ : } \vdash \Delta_{10}, \Delta_5, F_4, F_8 \& F_9} \text{ax} \quad \frac{\bullet h_7 \text{ : } \vdash \Delta_{10}, \text{dual}(F_6), F_8 \& F_9}{- \text{ : } \vdash \Delta_{10}, \Delta_5, F_3 \& F_4, F_8 \& F_9} \text{ax}}{- \text{ : } \vdash \Delta_{10}, \Delta_5, F_3 \& F_4, F_8 \& F_9} \text{hCut}}{\&}
\end{array}$$

- Case rule  $\oplus_B$

$$\begin{array}{c}
\frac{\frac{\frac{h_1 \text{ : } \vdash F_5, \Delta_2 \quad h_1 \text{ : } \vdash F_6, \Delta_2}{\bullet h_1 \text{ : } \vdash F_5 \& F_6, \Delta_2} \& \quad \frac{h_7 \text{ : } \vdash F_9, \Delta_{10}, \text{dual}(F_5) \oplus \text{dual}(F_6)}{\bullet h_7 \text{ : } \vdash \text{dual}(F_5 \& F_6), \Delta_{10}, F_8 \oplus F_9}}{- \text{ : } \vdash \Delta_2, \Delta_{10}, F_8 \oplus F_9} \text{Cut}}{\rightarrow} \\
\frac{\frac{\bullet h_1 \text{ : } \vdash \Delta_2, F_5 \& F_6}{- \text{ : } \vdash \Delta_{10}, \Delta_2, F_9} \text{ax} \quad \frac{h_7 \text{ : } \vdash \Delta_{10}, F_9, \text{dual}(F_5) \oplus \text{dual}(F_6)}{- \text{ : } \vdash \Delta_{10}, \Delta_2, F_8 \oplus F_9} \text{ax}}{- \text{ : } \vdash \Delta_{10}, \Delta_2, F_8 \oplus F_9} \text{hCut}}{\oplus_B} \\
\frac{\frac{h_1 \text{ : } \vdash F_5, \Delta_2 \quad h_1 \text{ : } \vdash F_6, \Delta_2}{\bullet h_1 \text{ : } \vdash F_5 \& F_6, \Delta_2} \& \quad \frac{h_7 \text{ : } \vdash \Delta_8, \text{dual}(F_6)}{\bullet h_7 \text{ : } \vdash \text{dual}(F_5 \& F_6), \Delta_8}}{- \text{ : } \vdash \Delta_2, \Delta_8} \text{Cut}}{\rightarrow} \\
\frac{\frac{- \text{ : } \vdash \Delta_2, F_6}{- \text{ : } \vdash \Delta_2, \Delta_8} \text{ax} \quad \frac{- \text{ : } \vdash \Delta_8, \text{dual}(F_6)}{- \text{ : } \vdash \Delta_2, \Delta_8} \text{ax}}{- \text{ : } \vdash \Delta_2, \Delta_8} \text{sCut}}{\oplus_B}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \vdash F_6, F_3, \Delta_5 \quad h_2 : \vdash F_6, F_4, \Delta_5}{\bullet h_2 : \vdash F_6, \Delta_5, F_3 \& F_4} \& \frac{h_7 : \vdash F_9, \Delta_{10}, dual(F_6)}{\bullet h_7 : \vdash dual(F_6), \Delta_{10}, F_8 \oplus F_9} \oplus_B \\
\hline
- : \vdash (\Delta_5, F_3 \& F_4), \Delta_{10}, F_8 \oplus F_9 \\
\rightarrow \\
\frac{\frac{\bullet h_2 : \vdash \Delta_5, F_6, F_3 \& F_4}{- : \vdash \Delta_{10}, \Delta_5, F_9, F_3 \& F_4} \text{ax} \quad \frac{h_7 : \vdash \Delta_{10}, F_9, dual(F_6)}{- : \vdash \Delta_{10}, \Delta_5, F_9, F_3 \& F_4} \text{ax}}{- : \vdash \Delta_{10}, \Delta_5, F_3 \& F_4, F_8 \oplus F_9} \text{hCut} \oplus_B
\end{array}$$

- Case rule  $\oplus_A$

$$\begin{array}{c}
\frac{h_1 : \vdash F_5, \Delta_2 \quad h_1 : \vdash F_6, \Delta_2}{\bullet h_1 : \vdash F_5 \& F_6, \Delta_2} \& \frac{h_7 : \vdash F_8, \Delta_{10}, dual(F_5) \oplus dual(F_6)}{\bullet h_7 : \vdash dual(F_5 \& F_6), \Delta_{10}, F_8 \oplus F_9} \oplus_A \\
\hline
- : \vdash \Delta_2, \Delta_{10}, F_8 \oplus F_9 \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \vdash \Delta_2, F_5 \& F_6}{- : \vdash \Delta_{10}, \Delta_2, F_8} \text{ax} \quad \frac{h_7 : \vdash \Delta_{10}, F_8, dual(F_5) \oplus dual(F_6)}{- : \vdash \Delta_{10}, \Delta_2, F_8 \oplus F_9} \text{ax}}{- : \vdash \Delta_{10}, \Delta_2, F_8 \oplus F_9} \text{hCut} \oplus_A
\end{array}$$

$$\begin{array}{c}
\frac{h_1 : \vdash F_5, \Delta_2 \quad h_1 : \vdash F_6, \Delta_2}{\bullet h_1 : \vdash F_5 \& F_6, \Delta_2} \& \frac{h_7 : \vdash \Delta_8, dual(F_5)}{\bullet h_7 : \vdash dual(F_5 \& F_6), \Delta_8} \oplus_A \\
\hline
- : \vdash \Delta_2, \Delta_8 \\
\rightarrow \\
\frac{\frac{- : \vdash \Delta_2, F_5}{- : \vdash \Delta_8, dual(F_5)} \text{ax} \quad \frac{- : \vdash \Delta_8, dual(F_5)}{- : \vdash \Delta_2, \Delta_8} \text{ax}}{- : \vdash \Delta_2, \Delta_8} \text{sCut}
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \vdash F_6, F_3, \Delta_5 \quad h_2 : \vdash F_6, F_4, \Delta_5}{\bullet h_2 : \vdash F_6, \Delta_5, F_3 \& F_4} \& \frac{h_7 : \vdash F_8, \Delta_{10}, dual(F_6)}{\bullet h_7 : \vdash dual(F_6), \Delta_{10}, F_8 \oplus F_9} \oplus_A \\
\hline
- : \vdash (\Delta_5, F_3 \& F_4), \Delta_{10}, F_8 \oplus F_9 \\
\rightarrow \\
\frac{\frac{\bullet h_2 : \vdash \Delta_5, F_6, F_3 \& F_4}{- : \vdash \Delta_{10}, \Delta_5, F_8, F_3 \& F_4} \text{ax} \quad \frac{h_7 : \vdash \Delta_{10}, F_8, dual(F_6)}{- : \vdash \Delta_{10}, \Delta_5, F_8, F_3 \& F_4} \text{ax}}{- : \vdash \Delta_{10}, \Delta_5, F_3 \& F_4, F_8 \oplus F_9} \text{hCut} \oplus_A
\end{array}$$

- Case rule  $\perp$

$$\begin{array}{c}
\frac{h_1 : \vdash F_5, \Delta_2 \quad h_1 : \vdash F_6, \Delta_2}{\bullet h_1 : \vdash F_5 \& F_6, \Delta_2} \& \frac{h_7 : \vdash \Delta_8, dual(F_5) \oplus dual(F_6)}{\bullet h_7 : \vdash dual(F_5 \& F_6), \perp, \Delta_8} \perp \\
\hline
- : \vdash \Delta_2, \perp, \Delta_8 \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \vdash \Delta_2, F_5 \& F_6}{- : \vdash \Delta_2, \Delta_8} \text{ax} \quad \frac{h_7 : \vdash \Delta_8, dual(F_5) \oplus dual(F_6)}{- : \vdash \Delta_2, \Delta_8, \perp} \text{ax}}{- : \vdash \Delta_2, \Delta_8, \perp} \text{hCut} \perp
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \vdash F_6, F_3, \Delta_5 \quad h_2 : \vdash F_6, F_4, \Delta_5}{\bullet h_2 : \vdash F_6, \Delta_5, F_3 \& F_4} \& \frac{h_7 : \vdash \Delta_8, dual(F_6)}{\bullet h_7 : \vdash dual(F_6), \perp, \Delta_8} \perp \\
\hline
- : \vdash (\Delta_5, F_3 \& F_4), \perp, \Delta_8 \\
\rightarrow \\
\frac{\frac{\bullet h_2 : \vdash \Delta_5, F_6, F_3 \& F_4}{- : \vdash \Delta_5, \Delta_8, F_3 \& F_4} \text{ax} \quad \frac{h_7 : \vdash \Delta_8, dual(F_6)}{- : \vdash \Delta_5, \Delta_8, \perp, F_3 \& F_4} \text{ax}}{- : \vdash \Delta_5, \Delta_8, \perp, F_3 \& F_4} \text{hCut} \perp
\end{array}$$

- Case rule  $\top$

$$\begin{array}{c}
\frac{h_1 : \vdash F_5, \Delta_2 \quad h_1 : \vdash F_6, \Delta_2}{\bullet h_1 : \vdash F_5 \& F_6, \Delta_2} \& \frac{}{\bullet h_7 : \vdash dual(F_5 \& F_6), \top, \Delta_8} \top \\
\hline
- : \vdash \Delta_2, \top, \Delta_8 \\
\rightarrow \\
\frac{}{- : \vdash \Delta_2, \Delta_8, \top} \top
\end{array}$$

$$\frac{\frac{h_2 \vdash F_6, F_3, \Delta_5 \quad h_2 \vdash F_6, F_4, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \& F_4} \& \quad \frac{}{\bullet h_7 \vdash \text{dual}(F_6), \top, \Delta_8} \top}{\frac{}{- \vdash (\Delta_5, F_3 \& F_4), \top, \Delta_8} \text{Cut}} \rightarrow$$

$$\frac{}{- \vdash \Delta_5, \Delta_8, \top, F_3 \& F_4} \top$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{\frac{h_1 \vdash F_5, \Delta_2 \quad h_1 \vdash F_6, \Delta_2}{\bullet h_1 \vdash F_5 \& F_6, \Delta_2} \& \quad \frac{h_7 \vdash F_9, \Delta_{11}, \text{dual}(F_5) \oplus \text{dual}(F_6) \quad h_7 \vdash F_{10}, \Delta_8}{\bullet h_7 \vdash \text{dual}(F_5 \& F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}}{\frac{}{- \vdash \Delta_2, \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}} \rightarrow$$

$$\frac{\frac{\bullet h_1 \vdash \Delta_2, F_5 \& F_6}{- \vdash \Delta_{11}, \Delta_2, F_9} \text{ax} \quad \frac{h_7 \vdash \Delta_{11}, F_9, \text{dual}(F_5) \oplus \text{dual}(F_6)}{h\text{Cut}} \text{ax}}{\frac{}{- \vdash \Delta_{11}, \Delta_2, \Delta_8, F_9 \otimes F_{10}} \otimes} \text{ax} \quad \frac{}{- \vdash \Delta_8, F_{10}} \text{ax}}{\otimes}$$

$$\frac{h_1 \vdash F_5, \Delta_2 \quad h_1 \vdash F_6, \Delta_2}{\bullet h_1 \vdash F_5 \& F_6, \Delta_2} \& \quad \frac{h_7 \vdash F_9, \Delta_8 \quad h_7 \vdash F_{10}, \Delta_{11}, \text{dual}(F_5) \oplus \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_5 \& F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}}{\frac{}{- \vdash \Delta_2, \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}} \rightarrow$$

$$\frac{\frac{}{- \vdash \Delta_8, F_9} \text{ax} \quad \frac{\bullet h_1 \vdash \Delta_2, F_5 \& F_6 \quad h_7 \vdash \Delta_{11}, F_{10}, \text{dual}(F_5) \oplus \text{dual}(F_6)}{- \vdash \Delta_{11}, \Delta_2, F_{10}} \text{ax}}{\frac{}{- \vdash \Delta_{11}, \Delta_2, \Delta_8, F_9 \otimes F_{10}} \otimes} \text{ax} \quad \frac{}{- \vdash \Delta_{11}, \Delta_2, F_{10}} \text{hCut}}{\otimes}$$

$$\frac{h_2 \vdash F_6, F_3, \Delta_5 \quad h_2 \vdash F_6, F_4, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \& F_4} \& \quad \frac{h_7 \vdash F_9, \Delta_{11}, \text{dual}(F_6) \quad h_7 \vdash F_{10}, \Delta_8}{\bullet h_7 \vdash \text{dual}(F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}}{\frac{}{- \vdash (\Delta_5, F_3 \& F_4), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}} \rightarrow$$

$$\frac{\frac{\bullet h_2 \vdash \Delta_5, F_6, F_3 \& F_4}{- \vdash \Delta_{11}, \Delta_5, F_9, F_3 \& F_4} \text{ax} \quad \frac{h_7 \vdash \Delta_{11}, F_9, \text{dual}(F_6)}{h\text{Cut}} \text{ax}}{\frac{}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_3 \& F_4, F_9 \otimes F_{10}} \otimes} \text{ax} \quad \frac{}{- \vdash \Delta_8, F_{10}} \text{ax}}{\otimes}$$

$$\frac{h_2 \vdash F_6, F_3, \Delta_5 \quad h_2 \vdash F_6, F_4, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \& F_4} \& \quad \frac{h_7 \vdash F_9, \Delta_8 \quad h_7 \vdash F_{10}, \Delta_{11}, \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}}{\frac{}{- \vdash (\Delta_5, F_3 \& F_4), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}} \rightarrow$$

$$\frac{\frac{}{- \vdash \Delta_8, F_9} \text{ax} \quad \frac{\bullet h_2 \vdash \Delta_5, F_6, F_3 \& F_4 \quad h_7 \vdash \Delta_{11}, F_{10}, \text{dual}(F_6)}{- \vdash \Delta_{11}, \Delta_5, F_{10}, F_3 \& F_4} \text{ax}}{\frac{}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_3 \& F_4, F_9 \otimes F_{10}} \otimes} \text{ax} \quad \frac{}{- \vdash \Delta_{11}, \Delta_5, F_{10}, F_3 \& F_4} \text{hCut}}{\otimes}$$

#### 4.4 Status of $\oplus_B$ : OK

- Case rule 1

- Case rule  $\$$

$$\frac{\frac{h_1 \vdash F_6, \Delta_2}{\bullet h_1 \vdash F_5 \oplus F_6, \Delta_2} \oplus_B \quad \frac{h_7 \vdash F_8, F_9, \Delta_{10}, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 \vdash \text{dual}(F_5 \oplus F_6), \Delta_{10}, F_8 \$ F_9} \$}{\frac{}{- \vdash \Delta_2, \Delta_{10}, F_8 \$ F_9} \text{Cut}} \rightarrow$$

$$\frac{\frac{\bullet h_1 \vdash \Delta_2, F_5 \oplus F_6}{- \vdash \Delta_{10}, \Delta_2, F_8, F_9} \text{ax} \quad \frac{h_7 \vdash \Delta_{10}, F_8, F_9, \text{dual}(F_5) \& \text{dual}(F_6)}{h\text{Cut}} \text{ax}}{\frac{}{- \vdash \Delta_{10}, \Delta_2, F_8 \$ F_9} \$} \text{ax} \quad \frac{}{- \vdash \Delta_{10}, \Delta_2, F_8 \$ F_9} \text{hCut}}{\$}$$

$$\begin{array}{c}
\frac{h_2 \text{ :- } F_6, F_4, \Delta_5}{\bullet h_2 \text{ :- } F_6, \Delta_5, F_3 \oplus F_4} \oplus_B \quad \frac{h_7 \text{ :- } F_8, F_9, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_6), \Delta_{10}, F_8 \& F_9} \& \\
\hline
\text{Cut} \\
\text{-- :- } (\Delta_5, F_3 \oplus F_4), \Delta_{10}, F_8 \& F_9 \\
\rightarrow \\
\frac{\frac{\bullet h_2 \text{ :- } \Delta_5, F_6, F_3 \oplus F_4}{\text{-- :- } \Delta_{10}, \Delta_5, F_8, F_9, F_3 \oplus F_4} \text{ax}}{\text{-- :- } \Delta_{10}, \Delta_5, F_8 \& F_9, F_3 \oplus F_4} \text{hCut}}{\text{-- :- } \Delta_{10}, \Delta_5, F_8 \& F_9, F_3 \oplus F_4} \&
\end{array}$$

• Case rule  $\&$

$$\begin{array}{c}
\frac{h_1 \text{ :- } F_6, \Delta_2}{\bullet h_1 \text{ :- } F_5 \oplus F_6, \Delta_2} \oplus_B \quad \frac{h_7 \text{ :- } F_8, \Delta_{10}, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \Delta_{10}, F_8 \& F_9} \& \quad \frac{h_7 \text{ :- } F_9, \Delta_{10}, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \Delta_{10}, F_8 \& F_9} \& \\
\hline
\text{Cut} \\
\text{-- :- } \Delta_2, \Delta_{10}, F_8 \& F_9 \\
\rightarrow \\
\frac{\frac{\bullet h_1 \text{ :- } \Delta_2, F_5 \oplus F_6}{\text{-- :- } \Delta_{10}, \Delta_2, F_8} \text{ax}}{\text{-- :- } \Delta_{10}, \Delta_2, F_8} \text{hCut}}{\text{-- :- } \Delta_{10}, \Delta_2, F_8 \& F_9} \& \quad \frac{\frac{\bullet h_7 \text{ :- } \Delta_{10}, F_9, \text{dual}(F_5) \& \text{dual}(F_6)}{\text{-- :- } \Delta_{10}, \Delta_2, F_9} \text{ax}}{\text{-- :- } \Delta_{10}, \Delta_2, F_9} \text{hCut}}{\text{-- :- } \Delta_{10}, \Delta_2, F_8 \& F_9} \&
\end{array}$$

$$\begin{array}{c}
\frac{h_1 \text{ :- } F_6, \Delta_2}{\bullet h_1 \text{ :- } F_5 \oplus F_6, \Delta_2} \oplus_B \quad \frac{h_7 \text{ :- } \Delta_8, \text{dual}(F_5)}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \Delta_8} \& \quad \frac{h_7 \text{ :- } \Delta_8, \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \Delta_8} \& \\
\hline
\text{Cut} \\
\text{-- :- } \Delta_2, \Delta_8 \\
\rightarrow \\
\frac{\frac{\text{-- :- } \Delta_2, F_6}{\text{-- :- } \Delta_2, \Delta_8} \text{ax}}{\text{-- :- } \Delta_2, \Delta_8} \text{sCut}}{\text{-- :- } \Delta_2, \Delta_8} \&
\end{array}$$

$$\begin{array}{c}
\frac{h_2 \text{ :- } F_6, F_4, \Delta_5}{\bullet h_2 \text{ :- } F_6, \Delta_5, F_3 \oplus F_4} \oplus_B \quad \frac{h_7 \text{ :- } F_8, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_6), \Delta_{10}, F_8 \& F_9} \& \quad \frac{h_7 \text{ :- } F_9, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_6), \Delta_{10}, F_8 \& F_9} \& \\
\hline
\text{Cut} \\
\text{-- :- } (\Delta_5, F_3 \oplus F_4), \Delta_{10}, F_8 \& F_9 \\
\rightarrow \\
\frac{\frac{h_2 \text{ :- } \Delta_5, F_4, F_6}{\text{-- :- } \Delta_{10}, \Delta_5, F_4, F_8 \& F_9} \text{ax}}{\text{-- :- } \Delta_{10}, \Delta_5, F_8 \& F_9, F_3 \oplus F_4} \text{hCut}}{\text{-- :- } \Delta_{10}, \Delta_5, F_8 \& F_9, F_3 \oplus F_4} \oplus_B
\end{array}$$

• Case rule  $\oplus_B$

$$\begin{array}{c}
\frac{h_1 \text{ :- } F_6, \Delta_2}{\bullet h_1 \text{ :- } F_5 \oplus F_6, \Delta_2} \oplus_B \quad \frac{h_7 \text{ :- } F_9, \Delta_{10}, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \Delta_{10}, F_8 \oplus F_9} \oplus_B \\
\hline
\text{Cut} \\
\text{-- :- } \Delta_2, \Delta_{10}, F_8 \oplus F_9 \\
\rightarrow \\
\frac{\frac{\bullet h_1 \text{ :- } \Delta_2, F_5 \oplus F_6}{\text{-- :- } \Delta_{10}, \Delta_2, F_9} \text{ax}}{\text{-- :- } \Delta_{10}, \Delta_2, F_8 \oplus F_9} \text{hCut}}{\text{-- :- } \Delta_{10}, \Delta_2, F_8 \oplus F_9} \oplus_B
\end{array}$$

$$\begin{array}{c}
\frac{h_2 \text{ :- } F_6, F_4, \Delta_5}{\bullet h_2 \text{ :- } F_6, \Delta_5, F_3 \oplus F_4} \oplus_B \quad \frac{h_7 \text{ :- } F_9, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_6), \Delta_{10}, F_8 \oplus F_9} \oplus_B \\
\hline
\text{Cut} \\
\text{-- :- } (\Delta_5, F_3 \oplus F_4), \Delta_{10}, F_8 \oplus F_9 \\
\rightarrow \\
\frac{\frac{h_2 \text{ :- } \Delta_5, F_4, F_6}{\text{-- :- } \Delta_{10}, \Delta_5, F_4, F_8 \oplus F_9} \text{ax}}{\text{-- :- } \Delta_{10}, \Delta_5, F_3 \oplus F_4, F_8 \oplus F_9} \text{hCut}}{\text{-- :- } \Delta_{10}, \Delta_5, F_3 \oplus F_4, F_8 \oplus F_9} \oplus_B
\end{array}$$

• Case rule  $\oplus_A$

$$\begin{array}{c}
\frac{h_1 \text{ :- } F_6, \Delta_2}{\bullet h_1 \text{ :- } F_5 \oplus F_6, \Delta_2} \oplus_B \quad \frac{h_7 \text{ :- } F_8, \Delta_{10}, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \Delta_{10}, F_8 \oplus F_9} \oplus_A \\
\hline
\text{Cut} \\
\text{-- :- } \Delta_2, \Delta_{10}, F_8 \oplus F_9 \\
\rightarrow \\
\frac{\frac{\bullet h_1 \text{ :- } \Delta_2, F_5 \oplus F_6}{\text{-- :- } \Delta_{10}, \Delta_2, F_8} \text{ax}}{\text{-- :- } \Delta_{10}, \Delta_2, F_8} \text{hCut}}{\text{-- :- } \Delta_{10}, \Delta_2, F_8 \oplus F_9} \oplus_A
\end{array}$$

$$\begin{array}{c}
\frac{h_2 \text{ :- } F_6, F_4, \Delta_5}{\bullet h_2 \text{ :- } F_6, \Delta_5, F_3 \oplus F_4} \oplus_B \quad \frac{h_7 \text{ :- } F_8, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_6), \Delta_{10}, F_8 \oplus F_9} \oplus_A \\
\text{Cut} \\
\hline
- \text{ :- } (\Delta_5, F_3 \oplus F_4), \Delta_{10}, F_8 \oplus F_9 \\
\rightarrow \\
\frac{\frac{h_2 \text{ :- } \Delta_5, F_4, F_6}{- \text{ :- } \Delta_{10}, \Delta_5, F_4, F_8 \oplus F_9} \text{ ax} \quad \frac{\bullet h_7 \text{ :- } \Delta_{10}, \text{dual}(F_6), F_8 \oplus F_9}{- \text{ :- } \Delta_{10}, \Delta_5, F_3 \oplus F_4, F_8 \oplus F_9} \text{ ax}}{- \text{ :- } \Delta_{10}, \Delta_5, F_3 \oplus F_4, F_8 \oplus F_9} \text{ hCut} \\
\oplus_B \\
- \text{ :- } \Delta_{10}, \Delta_5, F_3 \oplus F_4, F_8 \oplus F_9
\end{array}$$

- Case rule  $\perp$

$$\begin{array}{c}
\frac{h_1 \text{ :- } F_6, \Delta_2}{\bullet h_1 \text{ :- } F_5 \oplus F_6, \Delta_2} \oplus_B \quad \frac{h_7 \text{ :- } \Delta_8, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \perp, \Delta_8} \perp \\
\text{Cut} \\
\hline
- \text{ :- } \Delta_2, \perp, \Delta_8 \\
\rightarrow \\
\frac{\frac{h_1 \text{ :- } \Delta_2, F_5 \oplus F_6}{- \text{ :- } \Delta_2, \Delta_8} \text{ ax} \quad \frac{h_7 \text{ :- } \Delta_8, \text{dual}(F_5) \& \text{dual}(F_6)}{- \text{ :- } \Delta_2, \Delta_8, \perp} \text{ ax}}{- \text{ :- } \Delta_2, \Delta_8, \perp} \text{ hCut} \\
\perp \\
- \text{ :- } \Delta_2, \Delta_8, \perp
\end{array}$$

$$\begin{array}{c}
\frac{h_2 \text{ :- } F_6, F_4, \Delta_5}{\bullet h_2 \text{ :- } F_6, \Delta_5, F_3 \oplus F_4} \oplus_B \quad \frac{h_7 \text{ :- } \Delta_8, \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_6), \perp, \Delta_8} \perp \\
\text{Cut} \\
\hline
- \text{ :- } (\Delta_5, F_3 \oplus F_4), \perp, \Delta_8 \\
\rightarrow \\
\frac{\frac{h_2 \text{ :- } \Delta_5, F_4, F_6}{- \text{ :- } \Delta_5, \Delta_8, F_4, \perp} \text{ ax} \quad \frac{\bullet h_7 \text{ :- } \Delta_8, \perp, \text{dual}(F_6)}{- \text{ :- } \Delta_5, \Delta_8, \perp, F_3 \oplus F_4} \text{ ax}}{- \text{ :- } \Delta_5, \Delta_8, \perp, F_3 \oplus F_4} \text{ hCut} \\
\oplus_B \\
- \text{ :- } \Delta_5, \Delta_8, \perp, F_3 \oplus F_4
\end{array}$$

- Case rule  $\top$

$$\begin{array}{c}
\frac{h_1 \text{ :- } F_6, \Delta_2}{\bullet h_1 \text{ :- } F_5 \oplus F_6, \Delta_2} \oplus_B \quad \frac{}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \top, \Delta_8} \top \\
\text{Cut} \\
\hline
- \text{ :- } \Delta_2, \top, \Delta_8 \\
\rightarrow \\
\frac{}{- \text{ :- } \Delta_2, \Delta_8, \top} \top
\end{array}$$

$$\begin{array}{c}
\frac{h_2 \text{ :- } F_6, F_4, \Delta_5}{\bullet h_2 \text{ :- } F_6, \Delta_5, F_3 \oplus F_4} \oplus_B \quad \frac{}{\bullet h_7 \text{ :- } \text{dual}(F_6), \top, \Delta_8} \top \\
\text{Cut} \\
\hline
- \text{ :- } (\Delta_5, F_3 \oplus F_4), \top, \Delta_8 \\
\rightarrow \\
\frac{}{- \text{ :- } \Delta_5, \Delta_8, \top, F_3 \oplus F_4} \top
\end{array}$$

- Case rule  $I$

- Case rule  $\otimes$

$$\begin{array}{c}
\frac{h_1 \text{ :- } F_6, \Delta_2}{\bullet h_1 \text{ :- } F_5 \oplus F_6, \Delta_2} \oplus_B \quad \frac{h_7 \text{ :- } F_9, \Delta_{11}, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \otimes \quad h_7 \text{ :- } F_{10}, \Delta_8 \\
\text{Cut} \\
\hline
- \text{ :- } \Delta_2, \Delta_8, \Delta_{11}, F_9 \otimes F_{10} \\
\rightarrow \\
\frac{\frac{h_1 \text{ :- } \Delta_2, F_5 \oplus F_6}{- \text{ :- } \Delta_{11}, \Delta_2, F_9} \text{ ax} \quad \frac{h_7 \text{ :- } \Delta_{11}, F_9, \text{dual}(F_5) \& \text{dual}(F_6)}{- \text{ :- } \Delta_{11}, \Delta_2, \Delta_8, F_9 \otimes F_{10}} \text{ ax}}{- \text{ :- } \Delta_{11}, \Delta_2, F_9} \text{ hCut} \quad \frac{}{- \text{ :- } \Delta_8, F_{10}} \text{ ax}}{- \text{ :- } \Delta_{11}, \Delta_2, \Delta_8, F_9 \otimes F_{10}} \otimes
\end{array}$$

$$\begin{array}{c}
\frac{h_1 \text{ :- } F_6, \Delta_2}{\bullet h_1 \text{ :- } F_5 \oplus F_6, \Delta_2} \oplus_B \quad \frac{h_7 \text{ :- } F_9, \Delta_8 \quad h_7 \text{ :- } F_{10}, \Delta_{11}, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \otimes \\
\text{Cut} \\
\hline
- \text{ :- } \Delta_2, \Delta_8, \Delta_{11}, F_9 \otimes F_{10} \\
\rightarrow \\
\frac{}{- \text{ :- } \Delta_8, F_9} \text{ ax} \quad \frac{\frac{h_1 \text{ :- } \Delta_2, F_5 \oplus F_6}{- \text{ :- } \Delta_{11}, \Delta_2, F_{10}} \text{ ax} \quad \frac{h_7 \text{ :- } \Delta_{11}, F_{10}, \text{dual}(F_5) \& \text{dual}(F_6)}{- \text{ :- } \Delta_{11}, \Delta_2, F_{10}} \text{ ax}}{- \text{ :- } \Delta_{11}, \Delta_2, F_{10}} \text{ hCut}}{- \text{ :- } \Delta_{11}, \Delta_2, \Delta_8, F_9 \otimes F_{10}} \otimes
\end{array}$$

$$\begin{array}{c}
\frac{h_2 \vdash F_6, F_4, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \oplus F_4} \oplus_B \quad \frac{h_7 \vdash F_9, \Delta_{11}, dual(F_6) \quad h_7 \vdash F_{10}, \Delta_8}{\bullet h_7 \vdash dual(F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \otimes \\
\hline
- \vdash (\Delta_5, F_3 \oplus F_4), \Delta_8, \Delta_{11}, F_9 \otimes F_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_2 \vdash \Delta_5, F_4, F_6}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_4, F_9 \otimes F_{10}} \text{ax} \quad \frac{\bullet h_7 \vdash \Delta_{11}, \Delta_8, dual(F_6), F_9 \otimes F_{10}}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_9 \otimes F_{10}, F_3 \oplus F_4} \text{ax}}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_9 \otimes F_{10}, F_3 \oplus F_4} \oplus_B \quad \text{hCut} \\
\hline
\frac{h_2 \vdash F_6, F_4, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \oplus F_4} \oplus_B \quad \frac{h_7 \vdash F_9, \Delta_8 \quad h_7 \vdash F_{10}, \Delta_{11}, dual(F_6)}{\bullet h_7 \vdash dual(F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \otimes \\
\hline
- \vdash (\Delta_5, F_3 \oplus F_4), \Delta_8, \Delta_{11}, F_9 \otimes F_{10} \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_2 \vdash \Delta_5, F_4, F_6}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_4, F_9 \otimes F_{10}} \text{ax} \quad \frac{\bullet h_7 \vdash \Delta_{11}, \Delta_8, dual(F_6), F_9 \otimes F_{10}}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_9 \otimes F_{10}, F_3 \oplus F_4} \text{ax}}{- \vdash \Delta_{11}, \Delta_5, \Delta_8, F_9 \otimes F_{10}, F_3 \oplus F_4} \oplus_B \quad \text{hCut} \\
\hline
\end{array}$$

#### 4.5 Status of $\oplus_A$ : OK

- Case rule 1
- Case rule \$

$$\begin{array}{c}
\frac{h_1 \vdash F_5, \Delta_2}{\bullet h_1 \vdash F_5 \oplus F_6, \Delta_2} \oplus_A \quad \frac{h_7 \vdash F_8, F_9, \Delta_{10}, dual(F_5) \& dual(F_6)}{\bullet h_7 \vdash dual(F_5 \oplus F_6), \Delta_{10}, F_8 \$ F_9} \$ \\
\hline
- \vdash \Delta_2, \Delta_{10}, F_8 \$ F_9 \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 \vdash \Delta_2, F_5 \oplus F_6}{- \vdash \Delta_{10}, \Delta_2, F_8, F_9} \text{ax} \quad \frac{h_7 \vdash \Delta_{10}, F_8, F_9, dual(F_5) \& dual(F_6)}{- \vdash \Delta_{10}, \Delta_2, F_8 \$ F_9} \text{ax}}{- \vdash \Delta_{10}, \Delta_2, F_8 \$ F_9} \$ \quad \text{hCut} \\
\hline
\frac{h_2 \vdash F_6, F_3, \Delta_5}{\bullet h_2 \vdash F_6, \Delta_5, F_3 \oplus F_4} \oplus_A \quad \frac{h_7 \vdash F_8, F_9, \Delta_{10}, dual(F_6)}{\bullet h_7 \vdash dual(F_6), \Delta_{10}, F_8 \$ F_9} \$ \\
\hline
- \vdash (\Delta_5, F_3 \oplus F_4), \Delta_{10}, F_8 \$ F_9 \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_2 \vdash \Delta_5, F_6, F_3 \oplus F_4}{- \vdash \Delta_{10}, \Delta_5, F_8, F_9, F_3 \oplus F_4} \text{ax} \quad \frac{h_7 \vdash \Delta_{10}, F_8, F_9, dual(F_6)}{- \vdash \Delta_{10}, \Delta_5, F_8 \$ F_9, F_3 \oplus F_4} \text{ax}}{- \vdash \Delta_{10}, \Delta_5, F_8 \$ F_9, F_3 \oplus F_4} \$ \quad \text{hCut} \\
\hline
\end{array}$$

- Case rule &

$$\begin{array}{c}
\frac{h_1 \vdash F_5, \Delta_2}{\bullet h_1 \vdash F_5 \oplus F_6, \Delta_2} \oplus_A \quad \frac{h_7 \vdash F_8, \Delta_{10}, dual(F_5) \& dual(F_6) \quad h_7 \vdash F_9, \Delta_{10}, dual(F_5) \& dual(F_6)}{\bullet h_7 \vdash dual(F_5 \oplus F_6), \Delta_{10}, F_8 \& F_9} \& \\
\hline
- \vdash \Delta_2, \Delta_{10}, F_8 \& F_9 \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 \vdash \Delta_2, F_5 \oplus F_6}{- \vdash \Delta_{10}, \Delta_2, F_8} \text{ax} \quad \frac{h_7 \vdash \Delta_{10}, F_8, dual(F_5) \& dual(F_6)}{- \vdash \Delta_{10}, \Delta_2, F_9} \text{ax}}{- \vdash \Delta_{10}, \Delta_2, F_8 \& F_9} \& \quad \text{hCut} \\
\hline
\frac{h_1 \vdash F_5, \Delta_2}{\bullet h_1 \vdash F_5 \oplus F_6, \Delta_2} \oplus_A \quad \frac{h_7 \vdash \Delta_8, dual(F_5) \quad h_7 \vdash \Delta_8, dual(F_6)}{\bullet h_7 \vdash dual(F_5 \oplus F_6), \Delta_8} \& \\
\hline
- \vdash \Delta_2, \Delta_8 \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_1 \vdash \Delta_2, F_5}{- \vdash \Delta_2, \Delta_8} \text{ax} \quad \frac{h_7 \vdash \Delta_8, dual(F_5)}{- \vdash \Delta_2, \Delta_8} \text{ax}}{- \vdash \Delta_2, \Delta_8} \text{sCut} \\
\hline
\end{array}$$



$$\begin{array}{c}
\frac{h_2 \text{ :- } F_6, F_3, \Delta_5}{\bullet h_2 \text{ :- } F_6, \Delta_5, F_3 \oplus F_4} \oplus_A \quad \frac{h_7 \text{ :- } F_8, \Delta_{10}, \text{dual}(F_6) \quad h_7 \text{ :- } F_9, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_6), \Delta_{10}, F_8 \& F_9} \text{Cut} \quad \& \\
\frac{}{- \text{ :- } (\Delta_5, F_3 \oplus F_4), \Delta_{10}, F_8 \& F_9} \rightarrow \\
\frac{\frac{h_2 \text{ :- } \Delta_5, F_3, F_6}{\bullet h_2 \text{ :- } \Delta_5, F_3, F_6} \text{ax} \quad \frac{h_7 \text{ :- } \Delta_{10}, \text{dual}(F_6), F_8 \& F_9}{\bullet h_7 \text{ :- } \Delta_{10}, \text{dual}(F_6), F_8 \& F_9} \text{ax}}{- \text{ :- } \Delta_{10}, \Delta_5, F_3, F_8 \& F_9} \text{hCut}}{- \text{ :- } \Delta_{10}, \Delta_5, F_8 \& F_9, F_3 \oplus F_4} \oplus_A
\end{array}$$

- Case rule  $\oplus_B$

$$\begin{array}{c}
\frac{h_1 \text{ :- } F_5, \Delta_2}{\bullet h_1 \text{ :- } F_5 \oplus F_6, \Delta_2} \oplus_A \quad \frac{h_7 \text{ :- } F_9, \Delta_{10}, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \Delta_{10}, F_8 \oplus F_9} \oplus_B \quad \text{Cut} \\
\frac{}{- \text{ :- } \Delta_2, \Delta_{10}, F_8 \oplus F_9} \rightarrow \\
\frac{\frac{\bullet h_1 \text{ :- } \Delta_2, F_5 \oplus F_6}{\bullet h_1 \text{ :- } \Delta_2, F_5 \oplus F_6} \text{ax} \quad \frac{h_7 \text{ :- } \Delta_{10}, F_9, \text{dual}(F_5) \& \text{dual}(F_6)}{h_7 \text{ :- } \Delta_{10}, F_9, \text{dual}(F_5) \& \text{dual}(F_6)} \text{ax}}{- \text{ :- } \Delta_{10}, \Delta_2, F_9} \text{hCut}}{- \text{ :- } \Delta_{10}, \Delta_2, F_8 \oplus F_9} \oplus_B \\
\frac{h_2 \text{ :- } F_6, F_3, \Delta_5}{\bullet h_2 \text{ :- } F_6, \Delta_5, F_3 \oplus F_4} \oplus_A \quad \frac{h_7 \text{ :- } F_9, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_6), \Delta_{10}, F_8 \oplus F_9} \oplus_B \quad \text{Cut} \\
\frac{}{- \text{ :- } (\Delta_5, F_3 \oplus F_4), \Delta_{10}, F_8 \oplus F_9} \rightarrow \\
\frac{\frac{\bullet h_2 \text{ :- } \Delta_5, F_6, F_3 \oplus F_4}{\bullet h_2 \text{ :- } \Delta_5, F_6, F_3 \oplus F_4} \text{ax} \quad \frac{h_7 \text{ :- } \Delta_{10}, F_9, \text{dual}(F_6)}{h_7 \text{ :- } \Delta_{10}, F_9, \text{dual}(F_6)} \text{ax}}{- \text{ :- } \Delta_{10}, \Delta_5, F_9, F_3 \oplus F_4} \text{hCut}}{- \text{ :- } \Delta_{10}, \Delta_5, F_3 \oplus F_4, F_8 \oplus F_9} \oplus_B
\end{array}$$

- Case rule  $\oplus_A$

$$\begin{array}{c}
\frac{h_1 \text{ :- } F_5, \Delta_2}{\bullet h_1 \text{ :- } F_5 \oplus F_6, \Delta_2} \oplus_A \quad \frac{h_7 \text{ :- } F_8, \Delta_{10}, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \Delta_{10}, F_8 \oplus F_9} \oplus_A \quad \text{Cut} \\
\frac{}{- \text{ :- } \Delta_2, \Delta_{10}, F_8 \oplus F_9} \rightarrow \\
\frac{\frac{\bullet h_1 \text{ :- } \Delta_2, F_5 \oplus F_6}{\bullet h_1 \text{ :- } \Delta_2, F_5 \oplus F_6} \text{ax} \quad \frac{h_7 \text{ :- } \Delta_{10}, F_8, \text{dual}(F_5) \& \text{dual}(F_6)}{h_7 \text{ :- } \Delta_{10}, F_8, \text{dual}(F_5) \& \text{dual}(F_6)} \text{ax}}{- \text{ :- } \Delta_{10}, \Delta_2, F_8} \text{hCut}}{- \text{ :- } \Delta_{10}, \Delta_2, F_8 \oplus F_9} \oplus_A \\
\frac{h_2 \text{ :- } F_6, F_3, \Delta_5}{\bullet h_2 \text{ :- } F_6, \Delta_5, F_3 \oplus F_4} \oplus_A \quad \frac{h_7 \text{ :- } F_8, \Delta_{10}, \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_6), \Delta_{10}, F_8 \oplus F_9} \oplus_A \quad \text{Cut} \\
\frac{}{- \text{ :- } (\Delta_5, F_3 \oplus F_4), \Delta_{10}, F_8 \oplus F_9} \rightarrow \\
\frac{\frac{h_2 \text{ :- } \Delta_5, F_3, F_6}{\bullet h_2 \text{ :- } \Delta_5, F_3, F_6} \text{ax} \quad \frac{\bullet h_7 \text{ :- } \Delta_{10}, \text{dual}(F_6), F_8 \oplus F_9}{\bullet h_7 \text{ :- } \Delta_{10}, \text{dual}(F_6), F_8 \oplus F_9} \text{ax}}{- \text{ :- } \Delta_{10}, \Delta_5, F_3, F_8 \oplus F_9} \text{hCut}}{- \text{ :- } \Delta_{10}, \Delta_5, F_3 \oplus F_4, F_8 \oplus F_9} \oplus_A
\end{array}$$

- Case rule  $\perp$

$$\begin{array}{c}
\frac{h_1 \text{ :- } F_5, \Delta_2}{\bullet h_1 \text{ :- } F_5 \oplus F_6, \Delta_2} \oplus_A \quad \frac{h_7 \text{ :- } \Delta_8, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_5 \oplus F_6), \perp, \Delta_8} \perp \quad \text{Cut} \\
\frac{}{- \text{ :- } \Delta_2, \perp, \Delta_8} \rightarrow \\
\frac{\frac{\bullet h_1 \text{ :- } \Delta_2, F_5 \oplus F_6}{\bullet h_1 \text{ :- } \Delta_2, F_5 \oplus F_6} \text{ax} \quad \frac{h_7 \text{ :- } \Delta_8, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 \text{ :- } \Delta_8, \text{dual}(F_5) \& \text{dual}(F_6)} \text{ax}}{- \text{ :- } \Delta_2, \Delta_8} \text{hCut}}{- \text{ :- } \Delta_2, \Delta_8, \perp} \perp \\
\frac{h_2 \text{ :- } F_6, F_3, \Delta_5}{\bullet h_2 \text{ :- } F_6, \Delta_5, F_3 \oplus F_4} \oplus_A \quad \frac{h_7 \text{ :- } \Delta_8, \text{dual}(F_6)}{\bullet h_7 \text{ :- } \text{dual}(F_6), \perp, \Delta_8} \perp \quad \text{Cut} \\
\frac{}{- \text{ :- } (\Delta_5, F_3 \oplus F_4), \perp, \Delta_8} \rightarrow \\
\frac{\frac{h_2 \text{ :- } \Delta_5, F_3, F_6}{\bullet h_2 \text{ :- } \Delta_5, F_3, F_6} \text{ax} \quad \frac{\bullet h_7 \text{ :- } \Delta_8, \perp, \text{dual}(F_6)}{\bullet h_7 \text{ :- } \Delta_8, \perp, \text{dual}(F_6)} \text{ax}}{- \text{ :- } \Delta_5, \Delta_8, F_3, \perp} \text{hCut}}{- \text{ :- } \Delta_5, \Delta_8, \perp, F_3 \oplus F_4} \oplus_A
\end{array}$$

- Case rule  $\top$

$$\frac{\frac{h_1 : \vdash F_5, \Delta_2}{\bullet h_1 : \vdash F_5 \oplus F_6, \Delta_2} \oplus_A \frac{}{\bullet h_7 : \vdash \text{dual}(F_5 \oplus F_6), \top, \Delta_8} \top}{- : \vdash \Delta_2, \top, \Delta_8} \text{Cut}}{\rightarrow} \top$$

$$\frac{}{- : \vdash \Delta_2, \Delta_8, \top} \top$$
  

$$\frac{\frac{h_2 : \vdash F_6, F_3, \Delta_5}{\bullet h_2 : \vdash F_6, \Delta_5, F_3 \oplus F_4} \oplus_A \frac{}{\bullet h_7 : \vdash \text{dual}(F_6), \top, \Delta_8} \top}{- : \vdash (\Delta_5, F_3 \oplus F_4), \top, \Delta_8} \text{Cut}}{\rightarrow} \top$$

$$\frac{}{- : \vdash \Delta_5, \Delta_8, \top, F_3 \oplus F_4} \top$$

- Case rule  $I$

- Case rule  $\otimes$

$$\frac{\frac{h_1 : \vdash F_5, \Delta_2}{\bullet h_1 : \vdash F_5 \oplus F_6, \Delta_2} \oplus_A \frac{h_7 : \vdash F_9, \Delta_{11}, \text{dual}(F_5) \& \text{dual}(F_6) \quad h_7 : \vdash F_{10}, \Delta_8}{\bullet h_7 : \vdash \text{dual}(F_5 \oplus F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \otimes}{- : \vdash \Delta_2, \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}}{\rightarrow} \otimes$$

$$\frac{\frac{\bullet h_1 : \vdash \Delta_2, F_5 \oplus F_6}{- : \vdash \Delta_{11}, \Delta_2, F_9} \text{ax} \quad \frac{h_7 : \vdash \Delta_{11}, F_9, \text{dual}(F_5) \& \text{dual}(F_6)}{- : \vdash \Delta_{11}, \Delta_2, \Delta_8, F_9 \otimes F_{10}} \text{hCut} \quad \frac{}{- : \vdash \Delta_8, F_{10}} \text{ax}}{\otimes} \otimes$$
  

$$\frac{\frac{h_1 : \vdash F_5, \Delta_2}{\bullet h_1 : \vdash F_5 \oplus F_6, \Delta_2} \oplus_A \frac{h_7 : \vdash F_9, \Delta_8 \quad h_7 : \vdash F_{10}, \Delta_{11}, \text{dual}(F_5) \& \text{dual}(F_6)}{\bullet h_7 : \vdash \text{dual}(F_5 \oplus F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \otimes}{- : \vdash \Delta_2, \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}}{\rightarrow} \otimes$$

$$\frac{\frac{}{- : \vdash \Delta_8, F_9} \text{ax} \quad \frac{\bullet h_1 : \vdash \Delta_2, F_5 \oplus F_6}{- : \vdash \Delta_{11}, \Delta_2, F_{10}} \text{ax} \quad \frac{h_7 : \vdash \Delta_{11}, F_{10}, \text{dual}(F_5) \& \text{dual}(F_6)}{- : \vdash \Delta_{11}, \Delta_2, F_{10}} \text{hCut}}{\otimes} \otimes$$
  

$$\frac{\frac{h_2 : \vdash F_6, F_3, \Delta_5}{\bullet h_2 : \vdash F_6, \Delta_5, F_3 \oplus F_4} \oplus_A \frac{h_7 : \vdash F_9, \Delta_{11}, \text{dual}(F_6) \quad h_7 : \vdash F_{10}, \Delta_8}{\bullet h_7 : \vdash \text{dual}(F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \otimes}{- : \vdash (\Delta_5, F_3 \oplus F_4), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}}{\rightarrow} \otimes$$

$$\frac{\frac{\bullet h_2 : \vdash \Delta_5, F_3, F_6}{- : \vdash \Delta_{11}, \Delta_5, \Delta_8, F_3, F_9 \otimes F_{10}} \text{ax} \quad \frac{\bullet h_7 : \vdash \Delta_{11}, \Delta_8, \text{dual}(F_6), F_9 \otimes F_{10}}{- : \vdash \Delta_{11}, \Delta_5, \Delta_8, F_9 \otimes F_{10}, F_3 \oplus F_4} \text{hCut}}{\oplus_A} \oplus_A$$
  

$$\frac{\frac{h_2 : \vdash F_6, F_3, \Delta_5}{\bullet h_2 : \vdash F_6, \Delta_5, F_3 \oplus F_4} \oplus_A \frac{h_7 : \vdash F_9, \Delta_8 \quad h_7 : \vdash F_{10}, \Delta_{11}, \text{dual}(F_6)}{\bullet h_7 : \vdash \text{dual}(F_6), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \otimes}{- : \vdash (\Delta_5, F_3 \oplus F_4), \Delta_8, \Delta_{11}, F_9 \otimes F_{10}} \text{Cut}}{\rightarrow} \otimes$$

$$\frac{\frac{\bullet h_2 : \vdash \Delta_5, F_3, F_6}{- : \vdash \Delta_{11}, \Delta_5, \Delta_8, F_3, F_9 \otimes F_{10}} \text{ax} \quad \frac{\bullet h_7 : \vdash \Delta_{11}, \Delta_8, \text{dual}(F_6), F_9 \otimes F_{10}}{- : \vdash \Delta_{11}, \Delta_5, \Delta_8, F_9 \otimes F_{10}, F_3 \oplus F_4} \text{hCut}}{\oplus_A} \oplus_A$$

## 4.6 Status of $\perp$ : OK

- Case rule 1

$$\frac{\frac{h_1 : \vdash \Delta_2}{\bullet h_1 : \vdash \perp, \Delta_2} \perp \quad \frac{}{\bullet h_3 : \vdash \text{dual}(\perp), *} 1}{- : \vdash \Delta_2, *} \text{Cut}}{\rightarrow} 1$$

$$\frac{}{- : \vdash \Delta_2} \text{ax}$$

- Case rule  $\$$

$$\begin{array}{c}
\frac{h_1 \vdash \Delta_2}{\bullet h_1 \vdash \perp, \Delta_2} \perp \quad \frac{h_3 \vdash \mathbf{1}, F_4, F_5, \Delta_6}{\bullet h_3 \vdash \mathit{dual}(\perp), \Delta_6, F_4 \& F_5} \$ \\
\hline
- \vdash \Delta_2, \Delta_6, F_4 \& F_5 \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_1 \vdash \Delta_2, \perp}{- \vdash \Delta_2, \Delta_6, F_4, F_5} \text{ax} \quad \frac{h_3 \vdash \mathbf{1}, \Delta_6, F_4, F_5}{\bullet h_3 \vdash \mathit{dual}(\perp), \Delta_6, F_4 \& F_5} \text{ax}}{- \vdash \Delta_2, \Delta_6, F_4 \& F_5} \text{hCut} \\
\hline
- \vdash \Delta_2, \Delta_6, F_4 \& F_5 \quad \$
\end{array}$$

$$\begin{array}{c}
\frac{h_2 \vdash F_4, \Delta_3}{\bullet h_2 \vdash F_4, \perp, \Delta_3} \perp \quad \frac{h_5 \vdash F_6, F_7, \Delta_8, \mathit{dual}(F_4)}{\bullet h_5 \vdash \mathit{dual}(F_4), \Delta_8, F_6 \& F_7} \$ \\
\hline
- \vdash (\perp, \Delta_3), \Delta_8, F_6 \& F_7 \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_2 \vdash \Delta_3, F_4, \perp}{- \vdash \Delta_3, \Delta_8, F_6, F_7, \perp} \text{ax} \quad \frac{h_5 \vdash F_6, F_7, \Delta_8, \mathit{dual}(F_4)}{\bullet h_5 \vdash \mathit{dual}(F_4), \Delta_8, F_6 \& F_7} \text{ax}}{- \vdash \Delta_3, \Delta_8, F_6, F_7, \perp} \text{hCut} \\
\hline
- \vdash \Delta_3, \Delta_8, \perp, F_6 \& F_7 \quad \$
\end{array}$$

- Case rule  $\&$

$$\begin{array}{c}
\frac{h_1 \vdash \Delta_2}{\bullet h_1 \vdash \perp, \Delta_2} \perp \quad \frac{h_3 \vdash \mathbf{1}, F_4, \Delta_6 \quad h_3 \vdash \mathbf{1}, F_5, \Delta_6}{\bullet h_3 \vdash \mathit{dual}(\perp), \Delta_6, F_4 \& F_5} \& \\
\hline
- \vdash \Delta_2, \Delta_6, F_4 \& F_5 \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_1 \vdash \Delta_2, \perp}{- \vdash \Delta_2, \Delta_6, F_4} \text{ax} \quad \frac{h_3 \vdash \mathbf{1}, \Delta_6, F_4}{\bullet h_3 \vdash \mathit{dual}(\perp), \Delta_6, F_4 \& F_5} \text{ax} \quad \frac{\bullet h_1 \vdash \Delta_2, \perp}{- \vdash \Delta_2, \Delta_6, F_5} \text{ax} \quad \frac{h_3 \vdash \mathbf{1}, \Delta_6, F_5}{\bullet h_3 \vdash \mathit{dual}(\perp), \Delta_6, F_4 \& F_5} \text{ax}}{- \vdash \Delta_2, \Delta_6, F_4 \& F_5} \& \\
\hline
- \vdash \Delta_2, \Delta_6, F_4 \& F_5
\end{array}$$

$$\begin{array}{c}
\frac{h_2 \vdash F_4, \Delta_3}{\bullet h_2 \vdash F_4, \perp, \Delta_3} \perp \quad \frac{h_5 \vdash F_6, \Delta_8, \mathit{dual}(F_4) \quad h_5 \vdash F_7, \Delta_8, \mathit{dual}(F_4)}{\bullet h_5 \vdash \mathit{dual}(F_4), \Delta_8, F_6 \& F_7} \& \\
\hline
- \vdash (\perp, \Delta_3), \Delta_8, F_6 \& F_7 \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{h_2 \vdash \Delta_3, F_4}{\bullet h_2 \vdash \mathit{dual}(F_4), \Delta_8, F_6 \& F_7} \text{ax} \quad \frac{h_5 \vdash F_6, \Delta_8, \mathit{dual}(F_4)}{\bullet h_5 \vdash \mathit{dual}(F_4), \Delta_8, F_6 \& F_7} \text{ax}}{- \vdash \Delta_3, \Delta_8, F_6 \& F_7} \text{hCut} \\
\hline
- \vdash \Delta_3, \Delta_8, \perp, F_6 \& F_7 \quad \perp
\end{array}$$

- Case rule  $\oplus_B$

$$\begin{array}{c}
\frac{h_1 \vdash \Delta_2}{\bullet h_1 \vdash \perp, \Delta_2} \perp \quad \frac{h_3 \vdash \mathbf{1}, F_5, \Delta_6}{\bullet h_3 \vdash \mathit{dual}(\perp), \Delta_6, F_4 \oplus F_5} \oplus_B \\
\hline
- \vdash \Delta_2, \Delta_6, F_4 \oplus F_5 \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_1 \vdash \Delta_2, \perp}{- \vdash \Delta_2, \Delta_6, F_5} \text{ax} \quad \frac{h_3 \vdash \mathbf{1}, \Delta_6, F_5}{\bullet h_3 \vdash \mathit{dual}(\perp), \Delta_6, F_4 \oplus F_5} \text{ax}}{- \vdash \Delta_2, \Delta_6, F_5} \text{hCut} \\
\hline
- \vdash \Delta_2, \Delta_6, F_4 \oplus F_5 \quad \oplus_B
\end{array}$$

$$\begin{array}{c}
\frac{h_2 \vdash F_4, \Delta_3}{\bullet h_2 \vdash F_4, \perp, \Delta_3} \perp \quad \frac{h_5 \vdash F_7, \Delta_8, \mathit{dual}(F_4)}{\bullet h_5 \vdash \mathit{dual}(F_4), \Delta_8, F_6 \oplus F_7} \oplus_B \\
\hline
- \vdash (\perp, \Delta_3), \Delta_8, F_6 \oplus F_7 \quad \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_2 \vdash \Delta_3, F_4, \perp}{- \vdash \Delta_3, \Delta_8, F_7, \perp} \text{ax} \quad \frac{h_5 \vdash F_7, \Delta_8, \mathit{dual}(F_4)}{\bullet h_5 \vdash \mathit{dual}(F_4), \Delta_8, F_6 \oplus F_7} \text{ax}}{- \vdash \Delta_3, \Delta_8, F_7, \perp} \text{hCut} \\
\hline
- \vdash \Delta_3, \Delta_8, \perp, F_6 \oplus F_7 \quad \oplus_B
\end{array}$$

- Case rule  $\oplus_A$

$$\begin{array}{c}
\frac{h_1 : \vdash \Delta_2}{\bullet h_1 : \vdash \perp, \Delta_2} \perp \quad \frac{h_3 : \vdash \mathbf{1}, F_4, \Delta_6}{\bullet h_3 : \vdash \mathit{dual}(\perp), \Delta_6, F_4 \oplus F_5} \oplus A \\
\hline
- : \vdash \Delta_2, \Delta_6, F_4 \oplus F_5 \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \vdash \Delta_2, \perp}{- : \vdash \Delta_2, \Delta_6, F_4} \text{ax} \quad \frac{h_3 : \vdash \mathbf{1}, \Delta_6, F_4}{- : \vdash \Delta_2, \Delta_6, F_4 \oplus F_5} \text{hCut}}{- : \vdash \Delta_2, \Delta_6, F_4 \oplus F_5} \oplus A \\
\hline
\frac{h_2 : \vdash F_4, \Delta_3}{\bullet h_2 : \vdash F_4, \perp, \Delta_3} \perp \quad \frac{h_5 : \vdash F_6, \Delta_8, \mathit{dual}(F_4)}{\bullet h_5 : \vdash \mathit{dual}(F_4), \Delta_8, F_6 \oplus F_7} \oplus A \\
\hline
- : \vdash (\perp, \Delta_3), \Delta_8, F_6 \oplus F_7 \\
\rightarrow \\
\frac{\frac{\bullet h_2 : \vdash \Delta_3, F_4, \perp}{- : \vdash \Delta_3, \Delta_8, F_6, \perp} \text{ax} \quad \frac{h_5 : \vdash \Delta_8, F_6, \mathit{dual}(F_4)}{- : \vdash \Delta_3, \Delta_8, \perp, F_6 \oplus F_7} \text{hCut}}{- : \vdash \Delta_3, \Delta_8, \perp, F_6 \oplus F_7} \oplus A
\end{array}$$

- Case rule  $\perp$

$$\begin{array}{c}
\frac{h_1 : \vdash \Delta_2}{\bullet h_1 : \vdash \perp, \Delta_2} \perp \quad \frac{h_3 : \vdash \mathbf{1}, \Delta_4}{\bullet h_3 : \vdash \mathit{dual}(\perp), \perp, \Delta_4} \perp \\
\hline
- : \vdash \Delta_2, \perp, \Delta_4 \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \vdash \Delta_2, \perp}{- : \vdash \Delta_2, \Delta_4} \text{ax} \quad \frac{h_3 : \vdash \mathbf{1}, \Delta_4}{- : \vdash \Delta_2, \Delta_4, \perp} \text{hCut}}{- : \vdash \Delta_2, \Delta_4, \perp} \perp \\
\hline
\frac{h_2 : \vdash F_4, \Delta_3}{\bullet h_2 : \vdash F_4, \perp, \Delta_3} \perp \quad \frac{h_5 : \vdash \Delta_6, \mathit{dual}(F_4)}{\bullet h_5 : \vdash \mathit{dual}(F_4), \perp, \Delta_6} \perp \\
\hline
- : \vdash (\perp, \Delta_3), \perp, \Delta_6 \\
\rightarrow \\
\frac{\frac{h_2 : \vdash \Delta_3, F_4}{- : \vdash \Delta_3, \Delta_6, \perp} \text{ax} \quad \frac{\bullet h_5 : \vdash \Delta_6, \perp, \mathit{dual}(F_4)}{- : \vdash \Delta_3, \Delta_6, \perp, \perp} \text{hCut}}{- : \vdash \Delta_3, \Delta_6, \perp, \perp} \perp
\end{array}$$

- Case rule  $\top$

$$\begin{array}{c}
\frac{h_1 : \vdash \Delta_2}{\bullet h_1 : \vdash \perp, \Delta_2} \perp \quad \frac{}{\bullet h_3 : \vdash \mathit{dual}(\perp), \top, \Delta_4} \top \\
\hline
- : \vdash \Delta_2, \top, \Delta_4 \\
\rightarrow \\
- : \vdash \Delta_2, \Delta_4, \top \quad \top \\
\hline
\frac{h_2 : \vdash F_4, \Delta_3}{\bullet h_2 : \vdash F_4, \perp, \Delta_3} \perp \quad \frac{}{\bullet h_5 : \vdash \mathit{dual}(F_4), \top, \Delta_6} \top \\
\hline
- : \vdash (\perp, \Delta_3), \top, \Delta_6 \\
\rightarrow \\
- : \vdash \Delta_3, \Delta_6, \perp, \top \quad \top
\end{array}$$

- Case rule  $I$

- Case rule  $\otimes$

$$\begin{array}{c}
\frac{h_1 : \vdash \Delta_2}{\bullet h_1 : \vdash \perp, \Delta_2} \perp \quad \frac{h_3 : \vdash \mathbf{1}, F_5, \Delta_7 \quad h_3 : \vdash F_6, \Delta_4}{\bullet h_3 : \vdash \mathit{dual}(\perp), \Delta_4, \Delta_7, F_5 \otimes F_6} \otimes \\
\hline
- : \vdash \Delta_2, \Delta_4, \Delta_7, F_5 \otimes F_6 \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \vdash \Delta_2, \perp}{- : \vdash \Delta_2, \Delta_7, F_5} \text{ax} \quad \frac{h_3 : \vdash \mathbf{1}, \Delta_7, F_5}{- : \vdash \Delta_4, F_6} \text{hCut}}{- : \vdash \Delta_2, \Delta_4, \Delta_7, F_5 \otimes F_6} \otimes
\end{array}$$

$$\begin{array}{c}
\frac{h_1 \vdash \Delta_2}{\bullet h_1 \vdash \perp, \Delta_2} \perp \quad \frac{h_3 \vdash F_5, \Delta_4 \quad h_3 \vdash \mathbf{1}, F_6, \Delta_7}{\bullet h_3 \vdash \mathit{dual}(\perp), \Delta_4, \Delta_7, F_5 \otimes F_6} \otimes}{- \vdash \Delta_2, \Delta_4, \Delta_7, F_5 \otimes F_6} \text{Cut} \\
\rightarrow \\
\frac{}{- \vdash \Delta_4, F_5} \text{ax} \quad \frac{\bullet h_1 \vdash \Delta_2, \perp}{- \vdash \Delta_2, \Delta_4, \Delta_7, F_5 \otimes F_6} \text{ax} \quad \frac{h_3 \vdash \mathbf{1}, \Delta_7, F_6}{- \vdash \Delta_2, \Delta_7, F_6} \text{ax}}{- \vdash \Delta_2, \Delta_4, \Delta_7, F_5 \otimes F_6} \text{hCut} \\
\otimes \\
\frac{h_2 \vdash F_4, \Delta_3}{\bullet h_2 \vdash F_4, \perp, \Delta_3} \perp \quad \frac{h_5 \vdash F_7, \Delta_9, \mathit{dual}(F_4) \quad h_5 \vdash F_8, \Delta_6}{\bullet h_5 \vdash \mathit{dual}(F_4), \Delta_6, \Delta_9, F_7 \otimes F_8} \otimes}{- \vdash (\perp, \Delta_3), \Delta_6, \Delta_9, F_7 \otimes F_8} \text{Cut} \\
\rightarrow \\
\frac{h_2 \vdash \Delta_3, F_4}{- \vdash \Delta_3, \Delta_6, \Delta_9, F_7 \otimes F_8} \text{ax} \quad \frac{\bullet h_5 \vdash \Delta_6, \Delta_9, \mathit{dual}(F_4), F_7 \otimes F_8}{- \vdash \Delta_3, \Delta_6, \Delta_9, \perp, F_7 \otimes F_8} \text{ax}}{- \vdash \Delta_3, \Delta_6, \Delta_9, F_7 \otimes F_8} \text{hCut} \\
\perp \\
\frac{h_2 \vdash F_4, \Delta_3}{\bullet h_2 \vdash F_4, \perp, \Delta_3} \perp \quad \frac{h_5 \vdash F_7, \Delta_6 \quad h_5 \vdash F_8, \Delta_9, \mathit{dual}(F_4)}{\bullet h_5 \vdash \mathit{dual}(F_4), \Delta_6, \Delta_9, F_7 \otimes F_8} \otimes}{- \vdash (\perp, \Delta_3), \Delta_6, \Delta_9, F_7 \otimes F_8} \text{Cut} \\
\rightarrow \\
\frac{h_2 \vdash \Delta_3, F_4}{- \vdash \Delta_3, \Delta_6, \Delta_9, F_7 \otimes F_8} \text{ax} \quad \frac{\bullet h_5 \vdash \Delta_6, \Delta_9, \mathit{dual}(F_4), F_7 \otimes F_8}{- \vdash \Delta_3, \Delta_6, \Delta_9, \perp, F_7 \otimes F_8} \text{ax}}{- \vdash \Delta_3, \Delta_6, \Delta_9, \perp, F_7 \otimes F_8} \text{hCut} \\
\perp
\end{array}$$

#### 4.7 Status of $\top$ : OK

- Case rule  $\mathbf{1}$
- Case rule  $\$$

$$\begin{array}{c}
\frac{}{\bullet h_1 \vdash \top, \Delta_2} \top \quad \frac{h_3 \vdash \mathbf{0}, F_4, F_5, \Delta_6}{\bullet h_3 \vdash \mathit{dual}(\top), \Delta_6, F_4 \$ F_5} \$}{- \vdash \Delta_2, \Delta_6, F_4 \$ F_5} \text{Cut} \\
\rightarrow \\
\frac{\bullet h_1 \vdash \Delta_2, \top}{- \vdash \Delta_2, \Delta_6, F_4, F_5} \text{ax} \quad \frac{h_3 \vdash \mathbf{0}, \Delta_6, F_4, F_5}{- \vdash \Delta_2, \Delta_6, F_4 \$ F_5} \text{ax}}{- \vdash \Delta_2, \Delta_6, F_4, F_5} \text{hCut} \\
\$ \\
\frac{}{\bullet h_2 \vdash F_4, \top, \Delta_3} \top \quad \frac{h_5 \vdash F_6, F_7, \Delta_8, \mathit{dual}(F_4)}{\bullet h_5 \vdash \mathit{dual}(F_4), \Delta_8, F_6 \$ F_7} \$}{- \vdash (\top, \Delta_3), \Delta_8, F_6 \$ F_7} \text{Cut} \\
\rightarrow \\
\frac{}{- \vdash \Delta_3, \Delta_8, \top, F_6 \$ F_7} \top
\end{array}$$

- Case rule  $\&$

$$\begin{array}{c}
\frac{}{\bullet h_1 \vdash \top, \Delta_2} \top \quad \frac{h_3 \vdash \mathbf{0}, F_4, \Delta_6 \quad h_3 \vdash \mathbf{0}, F_5, \Delta_6}{\bullet h_3 \vdash \mathit{dual}(\top), \Delta_6, F_4 \& F_5} \&}{- \vdash \Delta_2, \Delta_6, F_4 \& F_5} \text{Cut} \\
\rightarrow \\
\frac{\bullet h_1 \vdash \Delta_2, \top}{- \vdash \Delta_2, \Delta_6, F_4} \text{ax} \quad \frac{h_3 \vdash \mathbf{0}, \Delta_6, F_4}{- \vdash \Delta_2, \Delta_6, F_4 \& F_5} \text{ax}}{- \vdash \Delta_2, \Delta_6, F_4} \text{hCut} \\
\frac{\bullet h_1 \vdash \Delta_2, \top}{- \vdash \Delta_2, \Delta_6, F_5} \text{ax} \quad \frac{h_3 \vdash \mathbf{0}, \Delta_6, F_5}{- \vdash \Delta_2, \Delta_6, F_4 \& F_5} \text{ax}}{- \vdash \Delta_2, \Delta_6, F_5} \text{hCut} \\
\& \\
\frac{}{\bullet h_2 \vdash F_4, \top, \Delta_3} \top \quad \frac{h_5 \vdash F_6, \Delta_8, \mathit{dual}(F_4) \quad h_5 \vdash F_7, \Delta_8, \mathit{dual}(F_4)}{\bullet h_5 \vdash \mathit{dual}(F_4), \Delta_8, F_6 \& F_7} \&}{- \vdash (\top, \Delta_3), \Delta_8, F_6 \& F_7} \text{Cut} \\
\rightarrow \\
\frac{}{- \vdash \Delta_3, \Delta_8, \top, F_6 \& F_7} \top
\end{array}$$

- Case rule  $\oplus_B$

$$\begin{array}{c}
\frac{}{\bullet h_1 \vdash \top, \Delta_2} \top \quad \frac{h_3 \vdash \mathbf{0}, F_5, \Delta_6}{\bullet h_3 \vdash \text{dual}(\top), \Delta_6, F_4 \oplus F_5} \oplus_B \\
\hline
- \vdash \Delta_2, \Delta_6, F_4 \oplus F_5 \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{}{\bullet h_1 \vdash \Delta_2, \top} \top \quad \frac{h_3 \vdash \mathbf{0}, \Delta_6, F_5}{\bullet h_3 \vdash \text{dual}(\top), \Delta_6, F_4 \oplus F_5} \text{ax} \\
\hline
- \vdash \Delta_2, \Delta_6, F_5 \quad \text{hCut} \\
\hline
- \vdash \Delta_2, \Delta_6, F_4 \oplus F_5 \quad \oplus_B
\end{array} \\
\hline
\frac{}{\bullet h_2 \vdash F_4, \top, \Delta_3} \top \quad \frac{h_5 \vdash F_7, \Delta_8, \text{dual}(F_4)}{\bullet h_5 \vdash \text{dual}(F_4), \Delta_8, F_6 \oplus F_7} \oplus_B \\
\hline
- \vdash (\top, \Delta_3), \Delta_8, F_6 \oplus F_7 \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{}{\bullet h_2 \vdash F_4, \top, \Delta_3} \top \\
\hline
- \vdash \Delta_3, \Delta_8, \top, F_6 \oplus F_7 \quad \top
\end{array}
\end{array}$$

- Case rule  $\oplus_A$

$$\begin{array}{c}
\frac{}{\bullet h_1 \vdash \top, \Delta_2} \top \quad \frac{h_3 \vdash \mathbf{0}, F_4, \Delta_6}{\bullet h_3 \vdash \text{dual}(\top), \Delta_6, F_4 \oplus F_5} \oplus_A \\
\hline
- \vdash \Delta_2, \Delta_6, F_4 \oplus F_5 \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{}{\bullet h_1 \vdash \Delta_2, \top} \text{ax} \quad \frac{h_3 \vdash \mathbf{0}, \Delta_6, F_4}{\bullet h_3 \vdash \text{dual}(\top), \Delta_6, F_4 \oplus F_5} \text{ax} \\
\hline
- \vdash \Delta_2, \Delta_6, F_4 \quad \text{hCut} \\
\hline
- \vdash \Delta_2, \Delta_6, F_4 \oplus F_5 \quad \oplus_A
\end{array} \\
\hline
\frac{}{\bullet h_2 \vdash F_4, \top, \Delta_3} \top \quad \frac{h_5 \vdash F_6, \Delta_8, \text{dual}(F_4)}{\bullet h_5 \vdash \text{dual}(F_4), \Delta_8, F_6 \oplus F_7} \oplus_A \\
\hline
- \vdash (\top, \Delta_3), \Delta_8, F_6 \oplus F_7 \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{}{\bullet h_2 \vdash F_4, \top, \Delta_3} \top \\
\hline
- \vdash \Delta_3, \Delta_8, \top, F_6 \oplus F_7 \quad \top
\end{array}
\end{array}$$

- Case rule  $\perp$

$$\begin{array}{c}
\frac{}{\bullet h_1 \vdash \top, \Delta_2} \top \quad \frac{h_3 \vdash \mathbf{0}, \Delta_4}{\bullet h_3 \vdash \text{dual}(\top), \perp, \Delta_4} \perp \\
\hline
- \vdash \Delta_2, \perp, \Delta_4 \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{}{\bullet h_1 \vdash \Delta_2, \perp, \top} \top \quad \frac{h_3 \vdash \mathbf{0}, \Delta_4}{\bullet h_3 \vdash \text{dual}(\top), \perp, \Delta_4} \text{ax} \\
\hline
- \vdash \Delta_2, \Delta_4, \perp \quad \text{hCut}
\end{array} \\
\hline
\frac{}{\bullet h_2 \vdash F_4, \top, \Delta_3} \top \quad \frac{h_5 \vdash \Delta_6, \text{dual}(F_4)}{\bullet h_5 \vdash \text{dual}(F_4), \perp, \Delta_6} \perp \\
\hline
- \vdash (\top, \Delta_3), \perp, \Delta_6 \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{}{\bullet h_2 \vdash F_4, \top, \Delta_3} \top \\
\hline
- \vdash \Delta_3, \Delta_6, \perp, \top \quad \top
\end{array}
\end{array}$$

- Case rule  $\top$

$$\begin{array}{c}
\frac{}{\bullet h_1 \vdash \top, \Delta_2} \top \quad \frac{}{\bullet h_3 \vdash \text{dual}(\top), \top, \Delta_4} \top \\
\hline
- \vdash \Delta_2, \top, \Delta_4 \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{}{\bullet h_1 \vdash \top, \Delta_2} \top \\
\hline
- \vdash \Delta_2, \Delta_4, \top \quad \top
\end{array} \\
\hline
\frac{}{\bullet h_2 \vdash F_4, \top, \Delta_3} \top \quad \frac{}{\bullet h_5 \vdash \text{dual}(F_4), \top, \Delta_6} \top \\
\hline
- \vdash (\top, \Delta_3), \top, \Delta_6 \quad \text{Cut} \\
\hline
\begin{array}{c}
\frac{}{\bullet h_2 \vdash F_4, \top, \Delta_3} \top \\
\hline
- \vdash \Delta_3, \Delta_6, \top, \top \quad \top
\end{array}
\end{array}$$

- Case rule  $I$

- Case rule  $\otimes$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_1 : \vdash \top, \Delta_2} \top \quad \frac{h_3 : \vdash \mathbf{O}, F_5, \Delta_7 \quad h_3 : \vdash F_6, \Delta_4}{\bullet h_3 : \vdash \mathit{dual}(\top), \Delta_4, \Delta_7, F_5 \otimes F_6} \otimes}{- : \vdash \Delta_2, \Delta_4, \Delta_7, F_5 \otimes F_6} \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \vdash \Delta_2, \top}{- : \vdash \Delta_2, \Delta_7, F_5} \text{ax} \quad \frac{h_3 : \vdash \mathbf{O}, \Delta_7, F_5}{h\text{Cut} \quad - : \vdash \Delta_4, F_6} \text{ax}}{- : \vdash \Delta_2, \Delta_4, \Delta_7, F_5 \otimes F_6} \otimes \\
\\
\frac{\frac{}{\bullet h_1 : \vdash \top, \Delta_2} \top \quad \frac{h_3 : \vdash F_5, \Delta_4 \quad h_3 : \vdash \mathbf{O}, F_6, \Delta_7}{\bullet h_3 : \vdash \mathit{dual}(\top), \Delta_4, \Delta_7, F_5 \otimes F_6} \otimes}{- : \vdash \Delta_2, \Delta_4, \Delta_7, F_5 \otimes F_6} \text{Cut} \\
\rightarrow \\
\frac{\frac{- : \vdash \Delta_4, F_5}{- : \vdash \Delta_2, \Delta_4, \Delta_7, F_5 \otimes F_6} \text{ax} \quad \frac{\bullet h_1 : \vdash \Delta_2, \top \quad h_3 : \vdash \mathbf{O}, \Delta_7, F_6}{- : \vdash \Delta_2, \Delta_7, F_6} \text{hCut} \text{ax}}{- : \vdash \Delta_2, \Delta_4, \Delta_7, F_5 \otimes F_6} \otimes \\
\\
\frac{\frac{}{\bullet h_2 : \vdash F_4, \top, \Delta_3} \top \quad \frac{h_5 : \vdash F_7, \Delta_9, \mathit{dual}(F_4) \quad h_5 : \vdash F_8, \Delta_6}{\bullet h_5 : \vdash \mathit{dual}(F_4), \Delta_6, \Delta_9, F_7 \otimes F_8} \otimes}{- : \vdash (\top, \Delta_3), \Delta_6, \Delta_9, F_7 \otimes F_8} \text{Cut} \\
\rightarrow \\
\frac{- : \vdash \Delta_3, \Delta_6, \Delta_9, \top, F_7 \otimes F_8}{- : \vdash \Delta_3, \Delta_6, \Delta_9, \top, F_7 \otimes F_8} \top \\
\\
\frac{\frac{}{\bullet h_2 : \vdash F_4, \top, \Delta_3} \top \quad \frac{h_5 : \vdash F_7, \Delta_6 \quad h_5 : \vdash F_8, \Delta_9, \mathit{dual}(F_4)}{\bullet h_5 : \vdash \mathit{dual}(F_4), \Delta_6, \Delta_9, F_7 \otimes F_8} \otimes}{- : \vdash (\top, \Delta_3), \Delta_6, \Delta_9, F_7 \otimes F_8} \text{Cut} \\
\rightarrow \\
\frac{- : \vdash \Delta_3, \Delta_6, \Delta_9, \top, F_7 \otimes F_8}{- : \vdash \Delta_3, \Delta_6, \Delta_9, \top, F_7 \otimes F_8} \top
\end{array}$$

#### 4.8 Status of $I$ : OK

- Case rule 1
- Case rule  $\$$

$$\begin{array}{c}
\frac{\frac{}{\bullet h_1 : \vdash \hat{\ }(\mathbf{n}_3), p(\mathbf{n}_3)} I \quad \frac{h_4 : \vdash F_5, F_6, \Delta_7, p(\mathbf{n}_3)}{\bullet h_4 : \vdash \mathit{dual}(\hat{\ }(\mathbf{n}_3)), \Delta_7, F_5 \& F_6} \$}{- : \vdash p(\mathbf{n}_3), \Delta_7, F_5 \& F_6} \text{Cut} \\
\rightarrow \\
\frac{- : \vdash \Delta_7, F_5, F_6, p(\mathbf{n}_3)}{- : \vdash \Delta_7, p(\mathbf{n}_3), F_5 \& F_6} \text{ax} \quad \$ \\
\\
\frac{\frac{}{\bullet h_1 : \vdash p(\mathbf{n}_3), \hat{\ }(\mathbf{n}_3)} I \quad \frac{h_4 : \vdash F_5, F_6, \Delta_7, \hat{\ }(\mathbf{n}_3)}{\bullet h_4 : \vdash \mathit{dual}(p(\mathbf{n}_3)), \Delta_7, F_5 \& F_6} \$}{- : \vdash \hat{\ }(\mathbf{n}_3), \Delta_7, F_5 \& F_6} \text{Cut} \\
\rightarrow \\
\frac{- : \vdash \Delta_7, F_5, F_6, \hat{\ }(\mathbf{n}_3)}{- : \vdash \Delta_7, \hat{\ }(\mathbf{n}_3), F_5 \& F_6} \text{ax} \quad \$
\end{array}$$

- Case rule  $\&$

$$\frac{\frac{}{\bullet h_1 : \vdash \hat{\ }(\mathbf{n}_3), p(\mathbf{n}_3)} I \quad \frac{h_4 : \vdash F_5, \Delta_7, p(\mathbf{n}_3) \quad h_4 : \vdash F_6, \Delta_7, p(\mathbf{n}_3)}{\bullet h_4 : \vdash \mathit{dual}(\hat{\ }(\mathbf{n}_3)), \Delta_7, F_5 \& F_6} \&}{- : \vdash p(\mathbf{n}_3), \Delta_7, F_5 \& F_6} \text{Cut} \\
\rightarrow \\
\frac{- : \vdash \Delta_7, F_5, p(\mathbf{n}_3)}{- : \vdash \Delta_7, p(\mathbf{n}_3), F_5 \& F_6} \text{ax} \quad \frac{- : \vdash \Delta_7, F_6, p(\mathbf{n}_3)}{\&}$$

$$\begin{array}{c}
\frac{\bullet\mathbf{h}_1 \vdash p(\mathbf{n}_3), \hat{\mathbf{n}}_3}{\bullet\mathbf{h}_1 \vdash p(\mathbf{n}_3), \hat{\mathbf{n}}_3} I \quad \frac{\mathbf{h}_4 \vdash F_5, \Delta_7, \hat{\mathbf{n}}_3 \quad \mathbf{h}_4 \vdash F_6, \Delta_7, \hat{\mathbf{n}}_3}{\bullet\mathbf{h}_4 \vdash dual(p(\mathbf{n}_3)), \Delta_7, F_5 \& F_6} \& \\
\hline
- \vdash \hat{\mathbf{n}}_3, \Delta_7, F_5 \& F_6 \quad \text{Cut} \\
\rightarrow \\
\frac{- \vdash \Delta_7, F_5, \hat{\mathbf{n}}_3}{- \vdash \Delta_7, \hat{\mathbf{n}}_3, F_5 \& F_6} \text{ax} \quad \frac{- \vdash \Delta_7, F_6, \hat{\mathbf{n}}_3}{- \vdash \Delta_7, \hat{\mathbf{n}}_3, F_5 \& F_6} \&
\end{array}$$

- Case rule  $\oplus_B$

$$\begin{array}{c}
\frac{\bullet\mathbf{h}_1 \vdash \hat{\mathbf{n}}_3, p(\mathbf{n}_3)}{\bullet\mathbf{h}_1 \vdash \hat{\mathbf{n}}_3, p(\mathbf{n}_3)} I \quad \frac{\mathbf{h}_4 \vdash F_6, \Delta_7, p(\mathbf{n}_3)}{\bullet\mathbf{h}_4 \vdash dual(\hat{\mathbf{n}}_3), \Delta_7, F_5 \oplus F_6} \oplus_B \\
\hline
- \vdash p(\mathbf{n}_3), \Delta_7, F_5 \oplus F_6 \quad \text{Cut} \\
\rightarrow \\
\frac{- \vdash \Delta_7, F_6, p(\mathbf{n}_3)}{- \vdash \Delta_7, p(\mathbf{n}_3), F_5 \oplus F_6} \text{ax} \quad \oplus_B \\
\oplus_B
\end{array}$$

$$\begin{array}{c}
\frac{\bullet\mathbf{h}_1 \vdash p(\mathbf{n}_3), \hat{\mathbf{n}}_3}{\bullet\mathbf{h}_1 \vdash p(\mathbf{n}_3), \hat{\mathbf{n}}_3} I \quad \frac{\mathbf{h}_4 \vdash F_6, \Delta_7, \hat{\mathbf{n}}_3}{\bullet\mathbf{h}_4 \vdash dual(p(\mathbf{n}_3)), \Delta_7, F_5 \oplus F_6} \oplus_B \\
\hline
- \vdash \hat{\mathbf{n}}_3, \Delta_7, F_5 \oplus F_6 \quad \text{Cut} \\
\rightarrow \\
\frac{- \vdash \Delta_7, F_6, \hat{\mathbf{n}}_3}{- \vdash \Delta_7, \hat{\mathbf{n}}_3, F_5 \oplus F_6} \text{ax} \quad \oplus_B \\
\oplus_B
\end{array}$$

- Case rule  $\oplus_A$

$$\begin{array}{c}
\frac{\bullet\mathbf{h}_1 \vdash \hat{\mathbf{n}}_3, p(\mathbf{n}_3)}{\bullet\mathbf{h}_1 \vdash \hat{\mathbf{n}}_3, p(\mathbf{n}_3)} I \quad \frac{\mathbf{h}_4 \vdash F_5, \Delta_7, p(\mathbf{n}_3)}{\bullet\mathbf{h}_4 \vdash dual(\hat{\mathbf{n}}_3), \Delta_7, F_5 \oplus F_6} \oplus_A \\
\hline
- \vdash p(\mathbf{n}_3), \Delta_7, F_5 \oplus F_6 \quad \text{Cut} \\
\rightarrow \\
\frac{- \vdash \Delta_7, F_5, p(\mathbf{n}_3)}{- \vdash \Delta_7, p(\mathbf{n}_3), F_5 \oplus F_6} \text{ax} \quad \oplus_A \\
\oplus_A
\end{array}$$

$$\begin{array}{c}
\frac{\bullet\mathbf{h}_1 \vdash p(\mathbf{n}_3), \hat{\mathbf{n}}_3}{\bullet\mathbf{h}_1 \vdash p(\mathbf{n}_3), \hat{\mathbf{n}}_3} I \quad \frac{\mathbf{h}_4 \vdash F_5, \Delta_7, \hat{\mathbf{n}}_3}{\bullet\mathbf{h}_4 \vdash dual(p(\mathbf{n}_3)), \Delta_7, F_5 \oplus F_6} \oplus_A \\
\hline
- \vdash \hat{\mathbf{n}}_3, \Delta_7, F_5 \oplus F_6 \quad \text{Cut} \\
\rightarrow \\
\frac{- \vdash \Delta_7, F_5, \hat{\mathbf{n}}_3}{- \vdash \Delta_7, \hat{\mathbf{n}}_3, F_5 \oplus F_6} \text{ax} \quad \oplus_A \\
\oplus_A
\end{array}$$

- Case rule  $\perp$

$$\begin{array}{c}
\frac{\bullet\mathbf{h}_1 \vdash \hat{\mathbf{n}}_3, p(\mathbf{n}_3)}{\bullet\mathbf{h}_1 \vdash \hat{\mathbf{n}}_3, p(\mathbf{n}_3)} I \quad \frac{\mathbf{h}_4 \vdash \Delta_5, p(\mathbf{n}_3)}{\bullet\mathbf{h}_4 \vdash dual(\hat{\mathbf{n}}_3), \perp, \Delta_5} \perp \\
\hline
- \vdash p(\mathbf{n}_3), \perp, \Delta_5 \quad \text{Cut} \\
\rightarrow \\
\frac{- \vdash \Delta_5, p(\mathbf{n}_3)}{- \vdash \Delta_5, \perp, p(\mathbf{n}_3)} \text{ax} \quad \perp \\
\perp
\end{array}$$

$$\begin{array}{c}
\frac{\bullet\mathbf{h}_1 \vdash p(\mathbf{n}_3), \hat{\mathbf{n}}_3}{\bullet\mathbf{h}_1 \vdash p(\mathbf{n}_3), \hat{\mathbf{n}}_3} I \quad \frac{\mathbf{h}_4 \vdash \Delta_5, \hat{\mathbf{n}}_3}{\bullet\mathbf{h}_4 \vdash dual(p(\mathbf{n}_3)), \perp, \Delta_5} \perp \\
\hline
- \vdash \hat{\mathbf{n}}_3, \perp, \Delta_5 \quad \text{Cut} \\
\rightarrow \\
\frac{- \vdash \Delta_5, \hat{\mathbf{n}}_3}{- \vdash \Delta_5, \perp, \hat{\mathbf{n}}_3} \text{ax} \quad \perp \\
\perp
\end{array}$$

- Case rule  $\top$

$$\begin{array}{c}
\frac{\bullet\mathbf{h}_1 \vdash \hat{\mathbf{n}}_3, p(\mathbf{n}_3)}{\bullet\mathbf{h}_1 \vdash \hat{\mathbf{n}}_3, p(\mathbf{n}_3)} I \quad \frac{\bullet\mathbf{h}_4 \vdash dual(\hat{\mathbf{n}}_3), \top, \Delta_5}{\bullet\mathbf{h}_4 \vdash dual(\hat{\mathbf{n}}_3), \top, \Delta_5} \top \\
\hline
- \vdash p(\mathbf{n}_3), \top, \Delta_5 \quad \text{Cut} \\
\rightarrow \\
- \vdash \Delta_5, \top, p(\mathbf{n}_3) \quad \top
\end{array}$$









$$\begin{array}{c}
\frac{h_2 : \vdash F_7, F_4, \Delta_6 \quad h_2 : \vdash F_5, \Delta_3}{\bullet h_2 : \vdash F_7, \Delta_3, \Delta_6, F_4 \otimes F_5} \otimes \frac{h_8 : \vdash \Delta_9, \mathit{dual}(F_7)}{\bullet h_8 : \vdash \mathit{dual}(F_7), \perp, \Delta_9} \perp}{- : \vdash (\Delta_3, \Delta_6, F_4 \otimes F_5), \perp, \Delta_9} \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_2 : \vdash \Delta_3, \Delta_6, F_7, F_4 \otimes F_5}{- : \vdash \Delta_3, \Delta_6, \Delta_9, F_4 \otimes F_5} \text{ax} \quad \frac{h_8 : \vdash \Delta_9, \mathit{dual}(F_7)}{h_8 : \vdash \Delta_9, \mathit{dual}(F_7)} \text{ax}}{- : \vdash \Delta_3, \Delta_6, \Delta_9, \perp, F_4 \otimes F_5} \text{hCut} \perp}{- : \vdash \Delta_3, \Delta_6, \Delta_9, \perp, F_4 \otimes F_5} \perp
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \vdash F_4, \Delta_3 \quad h_2 : \vdash F_7, F_5, \Delta_6}{\bullet h_2 : \vdash F_7, \Delta_3, \Delta_6, F_4 \otimes F_5} \otimes \frac{h_8 : \vdash \Delta_9, \mathit{dual}(F_7)}{\bullet h_8 : \vdash \mathit{dual}(F_7), \perp, \Delta_9} \perp}{- : \vdash (\Delta_3, \Delta_6, F_4 \otimes F_5), \perp, \Delta_9} \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_2 : \vdash \Delta_3, \Delta_6, F_7, F_4 \otimes F_5}{- : \vdash \Delta_3, \Delta_6, \Delta_9, F_4 \otimes F_5} \text{ax} \quad \frac{h_8 : \vdash \Delta_9, \mathit{dual}(F_7)}{h_8 : \vdash \Delta_9, \mathit{dual}(F_7)} \text{ax}}{- : \vdash \Delta_3, \Delta_6, \Delta_9, \perp, F_4 \otimes F_5} \text{hCut} \perp}{- : \vdash \Delta_3, \Delta_6, \Delta_9, \perp, F_4 \otimes F_5} \perp
\end{array}$$

- Case rule  $\top$

$$\begin{array}{c}
\frac{h_1 : \vdash F_6, \Delta_2 \quad h_1 : \vdash F_7, \Delta_3}{\bullet h_1 : \vdash F_6 \otimes F_7, \Delta_2, \Delta_3} \otimes \frac{}{\bullet h_8 : \vdash \mathit{dual}(F_6 \otimes F_7), \top, \Delta_9} \top}{- : \vdash (\Delta_2, \Delta_3), \top, \Delta_9} \text{Cut} \\
\rightarrow \\
- : \vdash \Delta_2, \Delta_3, \Delta_9, \top \top
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \vdash F_7, F_4, \Delta_6 \quad h_2 : \vdash F_5, \Delta_3}{\bullet h_2 : \vdash F_7, \Delta_3, \Delta_6, F_4 \otimes F_5} \otimes \frac{}{\bullet h_8 : \vdash \mathit{dual}(F_7), \top, \Delta_9} \top}{- : \vdash (\Delta_3, \Delta_6, F_4 \otimes F_5), \top, \Delta_9} \text{Cut} \\
\rightarrow \\
- : \vdash \Delta_3, \Delta_6, \Delta_9, \top, F_4 \otimes F_5 \top
\end{array}$$

$$\begin{array}{c}
\frac{h_2 : \vdash F_4, \Delta_3 \quad h_2 : \vdash F_7, F_5, \Delta_6}{\bullet h_2 : \vdash F_7, \Delta_3, \Delta_6, F_4 \otimes F_5} \otimes \frac{}{\bullet h_8 : \vdash \mathit{dual}(F_7), \top, \Delta_9} \top}{- : \vdash (\Delta_3, \Delta_6, F_4 \otimes F_5), \top, \Delta_9} \text{Cut} \\
\rightarrow \\
- : \vdash \Delta_3, \Delta_6, \Delta_9, \top, F_4 \otimes F_5 \top
\end{array}$$

- Case rule  $I$

- Case rule  $\otimes$

$$\begin{array}{c}
\frac{h_1 : \vdash F_6, \Delta_2 \quad h_1 : \vdash F_7, \Delta_3}{\bullet h_1 : \vdash F_6 \otimes F_7, \Delta_2, \Delta_3} \otimes \frac{h_8 : \vdash F_{10}, \Delta_{12}, \mathit{dual}(F_6)\$dual(F_7) \quad h_8 : \vdash F_{11}, \Delta_9}{\bullet h_8 : \vdash \mathit{dual}(F_6 \otimes F_7), \Delta_9, \Delta_{12}, F_{10} \otimes F_{11}} \otimes}{- : \vdash (\Delta_2, \Delta_3), \Delta_9, \Delta_{12}, F_{10} \otimes F_{11}} \text{Cut} \\
\rightarrow \\
\frac{\frac{\bullet h_1 : \vdash \Delta_2, \Delta_3, F_6 \otimes F_7}{- : \vdash \Delta_{12}, \Delta_2, \Delta_3, F_{10}} \text{ax} \quad \frac{h_8 : \vdash \Delta_{12}, F_{10}, \mathit{dual}(F_6)\$dual(F_7)}{h_8 : \vdash \Delta_{12}, F_{10}, \mathit{dual}(F_6)\$dual(F_7)} \text{ax}}{- : \vdash \Delta_{12}, \Delta_2, \Delta_3, \Delta_9, F_{10} \otimes F_{11}} \text{hCut} \quad \frac{}{- : \vdash \Delta_9, F_{11}} \text{ax}}{- : \vdash \Delta_{12}, \Delta_2, \Delta_3, \Delta_9, F_{10} \otimes F_{11}} \otimes
\end{array}$$

$$\begin{array}{c}
\frac{h_1 : \vdash F_6, \Delta_2 \quad h_1 : \vdash F_7, \Delta_3}{\bullet h_1 : \vdash F_6 \otimes F_7, \Delta_2, \Delta_3} \otimes \frac{h_8 : \vdash F_{10}, \Delta_9 \quad h_8 : \vdash F_{11}, \Delta_{12}, \mathit{dual}(F_6)\$dual(F_7)}{\bullet h_8 : \vdash \mathit{dual}(F_6 \otimes F_7), \Delta_9, \Delta_{12}, F_{10} \otimes F_{11}} \otimes}{- : \vdash (\Delta_2, \Delta_3), \Delta_9, \Delta_{12}, F_{10} \otimes F_{11}} \text{Cut} \\
\rightarrow \\
\frac{}{- : \vdash \Delta_9, F_{10}} \text{ax} \quad \frac{\frac{\bullet h_1 : \vdash \Delta_2, \Delta_3, F_6 \otimes F_7}{- : \vdash \Delta_{12}, \Delta_2, \Delta_3, F_{11}} \text{ax} \quad \frac{h_8 : \vdash \Delta_{12}, F_{11}, \mathit{dual}(F_6)\$dual(F_7)}{h_8 : \vdash \Delta_{12}, F_{11}, \mathit{dual}(F_6)\$dual(F_7)} \text{ax}}{- : \vdash \Delta_{12}, \Delta_2, \Delta_3, F_{11}} \text{hCut} \quad \otimes}{- : \vdash \Delta_{12}, \Delta_2, \Delta_3, \Delta_9, F_{10} \otimes F_{11}} \otimes
\end{array}$$

